EDEXCEL HIGHER

FLUID MECHANICS H1 UNIT 8

NQF LEVEL 4

OUTCOME 1 STATIC FLUID SYSTEMS

TUTORIAL 1 - HYDROSTATICS

1 **Static fluid systems**
Immersed surfaces**:** rectangular and circular surfaces (eg retaining walls, tank sides, sluice gates, inspection covers, valve flanges)
C***entre of pressure*****:** use of parallel axis theorem for immersed rectangular and circular immersed surfaces
Devices**:** hydraulic presses; hydraulic jacks; hydraulic accumulators; braking systems; determine outputs for given inputs

On completion of this tutorial you should be able to do the following.

- Define the main ***fundamental properties*** of liquids.
- Calculate the forces and moments on ***submerged surfaces.***
- Explain and solve problems involving simple ***hydrostatic devices.***

Before you start you should make sure that you fully understand first and second moments of area. If you are not familiar with this, you should do that tutorial before proceeding. Let's start this tutorial by studying the fundamental properties of liquids.

1. SOME FUNDAMENTAL STUDIES

1.1 IDEAL LIQUIDS

An ideal liquid is defined as follows.

It is ***INVISCID***. This means that molecules require no force to separate them. The topic is covered in detail in chapter 3.

It is ***INCOMPRESSIBLE***. This means that it would require an infinite force to reduce the volume of the liquid.

1.2 REAL LIQUIDS

VISCOSITY

Real liquids have ***VISCOSITY***. This means that the molecules tend to stick to each other and to any surface with which they come into contact. This produces fluid friction and energy loss when the liquid flows over a surface. Viscosity defines how easily a liquid flows. The lower the viscosity, the easier it flows.

BULK MODULUS

Real liquids are compressible and this is governed by the ***BULK MODULUS K***. This is defined as follows.

$$K = V\Delta p/\Delta V$$

Δp is the increase in pressure, ΔV is the reduction in volume and V is the original volume.

DENSITY Density ρ relates the mass and volume such that $\rho = m/V$ kg/ m^3

PRESSURE

Pressure is the result of compacting the molecules of a fluid into a smaller space than it would otherwise occupy. Pressure is the force per unit area acting on a surface. The unit of pressure is the N/m^2 and this is called a ***PASCAL***. The Pascal is a small unit of pressure so higher multiples are common.

1 kPa = 10^3 N/m^2
1 MPa = 10^6 N/m^2
Another common unit of pressure is the ***bar*** but this is not an SI unit.
1 bar = 10^5 N/m^2
1 mb = 100 N/m^2

GAUGE AND ABSOLUTE PRESSURE

Most pressure gauges are designed only to measure and indicate the pressure of a fluid above that of the surrounding atmosphere and indicate zero when connected to the atmosphere. These are called ***gauge pressures*** and are normally used. Sometimes it is necessary to add the atmospheric pressure onto the gauge reading in order to find the true or ***absolute pressure***.

Absolute pressure = gauge pressure + atmospheric pressure.

Standard atmospheric pressure is 1.013 bar.

2. HYDROSTATIC FORCES

When you have completed this section, you should be able to do the following.

- Calculate the pressure due to the depth of a liquid.
- Calculate the total force on a vertical surface.
- Define and calculate the position of the centre of pressure for various shapes.
- Calculate the turning moments produced on vertically immersed surfaces.
- Explain the principles of simple hydraulic devices.
- Calculate the force and movement produced by simple hydraulic equipment.

2.1 HYDROSTATIC PRESSURE

2.1.1 PRESSURE INSIDE PIPES AND VESSELS

Pressure results when a liquid is compacted into a volume. The pressure inside vessels and pipes produce stresses and strains as it tries to stretch the material. An example of this is a pipe with flanged joints. The pressure in the pipe tries to separate the flanges. The force is the product of the pressure and the bore area.

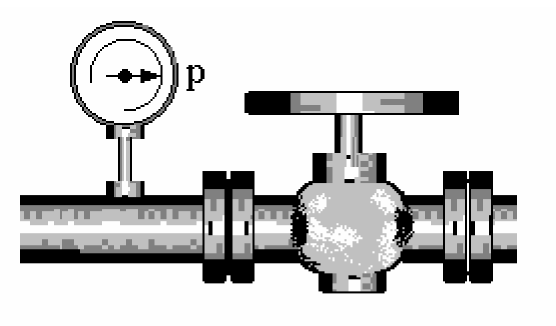


Fig.1

WORKED EXAMPLE No. 1

Calculate the force trying to separate the flanges of a valve (Fig.1) when the pressure is 2 MPa and the pipe bore is 50 mm.

SOLUTION

Force = pressure x bore area
Bore area = $\pi D^2/4 = \pi \times 0.05^2/4 = 1.963 \times 10^{-3}\ m^2$
Pressure = 2×10^6 Pa
Force = $2 \times 10^6 \times 1.963 \times 10^{-3} = 3.927 \times 10^3$ N or ***3.927 kN***

2.1.2 PRESSURE DUE TO THE WEIGHT OF A LIQUID

Consider a tank full of liquid as shown. The liquid has a total weight W and this bears down on the bottom and produces a pressure p. Pascal showed that the pressure in a liquid always acts normal (at 90^o) to the surface of contact so the pressure pushes down onto the bottom of the tank. He also showed that the pressure at a given point acts equally in all directions so the pressure also pushes up on the liquid above it and sideways against the walls.

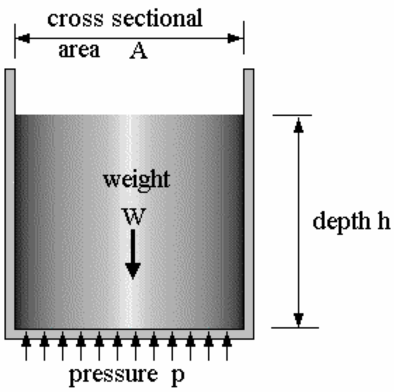


Fig. 2

The volume of the liquid is $V = A\,h$ m^3

The mass of liquid is hence $m = \rho V = \rho A h$ kg
The weight is obtained by multiplying by the gravitational constant g.
$W = mg = \rho A h g$ Newton

The pressure on the bottom is the weight per unit area $p = W/A$ N/m^2

It follows that the pressure at a depth h in a liquid is given by the following equation.

$$p = \rho g h$$

The unit of pressure is the N/m^2 and this is called a ***PASCAL***. The Pascal is a small unit of pressure so higher multiples are common.

WORKED EXAMPLE 2

Calculate the pressure and force on an inspection hatch 0.75 m diameter located on the bottom of a tank when it is filled with oil of density 875 kg/m^3 to a depth of 7 m.

SOLUTION

The pressure on the bottom of the tank is found as follows. $p = \rho\, g\, h$
$\rho =$ 875 kg/m^3
g = 9.81 m/s2
h = 7 m
$p = 875 \times 9.81 \times 7 = 60086$ N/m^2 or ***60.086 kPa***
The force is the product of pressure and area.
$A = \pi D^2/4 = \pi \times 0.75^2/4 = 0.442$ m^2
$F = p\,A = 60.086 \times 10^3 \times 0.442 = 26.55 \times 10^3$ N or ***26.55 Kn***

2.1.3 PRESSURE HEAD

When h is made the subject of the formula, it is called the pressure head. $\boldsymbol{h = p/\rho g}$

Pressure is often measured by using a column of liquid. Consider a pipe carrying liquid at pressure p. If a small vertical pipe is attached to it, the liquid will rise to a height h and at this height, the pressure at the foot of the column is equal to the pressure in the pipe.

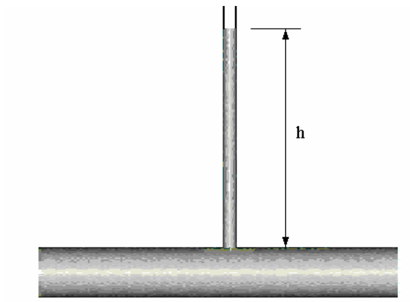
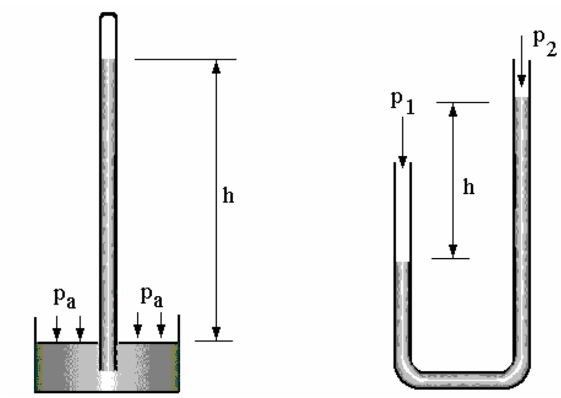


Fig.3

This principle is used in barometers to measure atmospheric pressure and manometers to measure gas pressure.



Barometer Manometer

Fig.4

In the manometer, the weight of the gas is negligible so the height h represents the difference in the pressures p_1 and p_2.

$$p_1 - p_2 = \rho g h$$

In the case of the barometer, the column is closed at the top so that $p_2 = 0$ and $p_1 = p_a$. The height h represents the atmospheric pressure. Mercury is used as the liquid because it does not evaporate easily at the near total vacuum on the top of the column.

$$P_a = \rho g h$$

WORKED EXAMPLE No.3

A manometer (fig.4) is used to measure the pressure of gas in a container. One side is connected to the container and the other side is open to the atmosphere. The manometer contains oil of density 750 kg/m^3 and the head is 50 mm. Calculate the gauge pressure of the gas in the container.

SOLUTION

$p_1 - p_2 = \rho g h = 750 \times 9.81 \times 0.05 = 367.9$ Pa

Since p_2 is atmospheric pressure, this is the gauge pressure. $\boldsymbol{p_2 = 367.9}$ ***Pa (gauge)***

SELF ASSESSMENT EXERCISE No.1

1. A mercury barometer gives a pressure head of 758 mm. The density is 13 600 kg/m3. Calculate the atmospheric pressure in bar. (1.0113 bar)

2. A manometer (fig.4) is used to measure the pressure of gas in a container. One side is connected to the container and the other side is open to the atmosphere. The manometer contains water of density 1000 kg/m^3 and the head is 250 mm. Calculate the gauge pressure of the gas in the container. (2.452.5 kPa)

3. Calculate the pressure and force on a horizontal submarine hatch 1.2 m diameter when it is at a depth of 800 m in seawater of density 1030 kg/m^3. (8.083 MPa and 9.142 MN)

3. FORCES ON SUBMERGED SURFACES

3.1 TOTAL FORCE

Consider a vertical area submerged below the surface of liquid as shown.

The area of the elementary strip is dA = B dy

You should already know that the pressure at depth h in a liquid is given by the equation p = ρgh where ρ is the density and h the depth.

In this case, we are using y to denote depth so p = ρgy

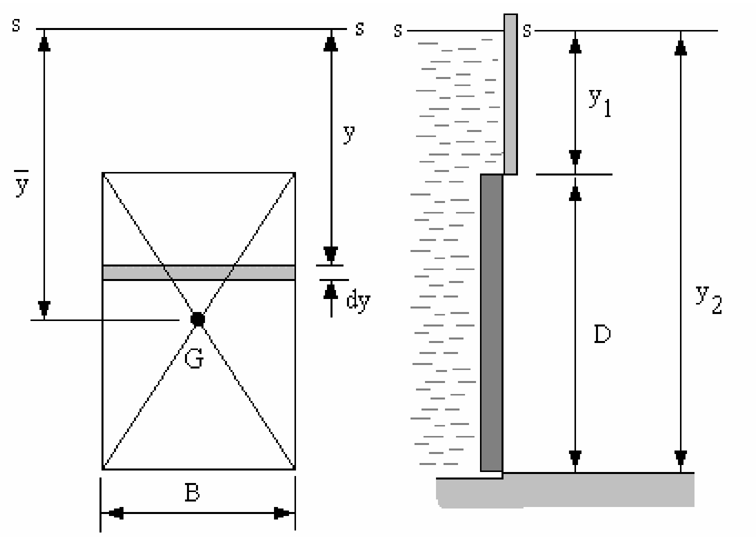


Fig.5

The force on the strip due to this pressure is $dF = p\,dA = \rho B\,gy\,dy$

The total force on the surface due to pressure is denoted R and it is obtained by integrating this expression between the limits of y_1 and y_2.

It follows that $$R = \rho gB\left(\frac{y_2^2 - y_1^2}{2}\right)$$

This may be factorised. $$R = \rho gB\frac{(y_2 - y_1)(y_2 + y_1)}{2}$$

$(y_2 - y_1) = D$ so $B(y_2 - y_1) = BD$ = Area of the surface A

$(y_2 + y_1)/2$ is the distance from the free surface to the centroid y.

It follows that the total force is given by the expression

$$R = \rho gAy$$

The term Ay is the first moment of area and in general, the total force on a submerged surface is

R = ρg x 1st moment of area about the free surface.

3.2 CENTRE OF PRESSURE

The centre of pressure is the point at which the total force may be assumed to act on a submerged surface. Consider the diagram again. The force on the strip is dF as before. This force produces a turning moment with respect to the free surface s – s. The turning moment due to dF is as follows.

$$dM = y\ dF = \rho g B y^2 dy$$

The total turning moment about the surface due to pressure is obtained by integrating this expression between the limits of y_1 and y_2.

$$M = \int_{y_2}^{y_2} \rho g B y^2 dy = \rho g B \int_{y_2}^{y_2} y^2 dy$$

$$\text{By definition } I_{ss} = B \int_{y_2}^{y_2} y^2 dy$$

Hence $\quad$ $\boldsymbol{M = \rho g I_{SS}}$

This moment must also be given by the total force R multiplied by some distance h. The position at depth h is called the **CENTRE OF PRESSURE**. h is found by equating the moments.

$$M = h\,R = h\,\rho\,g\,A\,\bar{y} = \rho\,g\,I_{ss}$$

$$\bar{h} = \frac{\rho\,g\,I_{ss}}{\rho\,g\,A\,\bar{y}} = \frac{I_{ss}}{A\,\bar{y}}$$

$$\bar{h} = \frac{2^{nd}\ \text{moment of area}}{1^{st}\ \text{moment of area}}\ \text{about s - s}$$

In order to be competent in this work, you should be familiar with the parallel axis theorem (covered in part 1) because you will need it to solve 2^{nd} moments of area about the free surface. The rule is as follows.

$$\mathbf{I_{SS} = I_{gg} + Ay^2}$$

$\mathbf{I_{SS}}$ is the 2^{nd} moment about the free surface and $\mathbf{I_{gg}}$ is the 2^{nd} moment about the centroid.

You should be familiar with the following standard formulae for 2^{nd} moments about the centroid.

Rectangle $\quad I_{gg} = BD^3/12$

Rectangle about its edge $\quad I = BD^3/3$

Circle $\quad I_{gg} = \pi D^4/64$

WORKED EXAMPLE No.4

Show that the centre of pressure on a vertical retaining wall is at 2/3 of the depth. Go on to show that the turning moment produced about the bottom of the wall is given by the expression $\rho gh^3/6$ for a unit width.

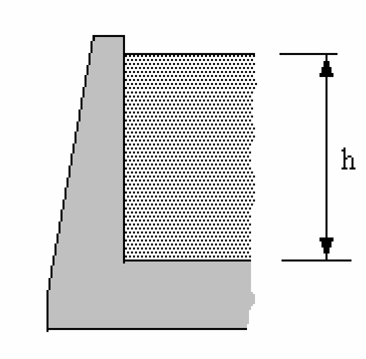


Fig.6

SOLUTION

For a given width B, the area is a rectangle with the free surface at the top edge.

$$\bar{y} = \frac{h}{2} \qquad A = bh$$

1^{st} moment of area about the top edge is $A\bar{y} = B\frac{h^2}{2}$

2^{nd} moment of area about the top edge is $B\frac{h^3}{3}$

$$\bar{h} = \frac{2^{nd}\text{ moment}}{1^{st}\text{ moment}} = \frac{B\frac{h^3}{3}}{B\frac{h^2}{2}}$$

$$\bar{h} = \frac{2h}{3}$$

It follows that the centre of pressure is h/3 from the bottom.

The total force is $R = \rho gAy = \rho gBh^2/2$ and for a unit width this is $\rho gh^2/2$

The moment bout the bottom is $R \times h/3 = (\rho gh^2/2) \times h/3 = \boldsymbol{\rho gh^3/6}$

SELF ASSESSMENT EXERCISE No.2

1. A vertical retaining wall contains water to a depth of 20 metres. Calculate the turning moment about the bottom for a unit width. Take the density as 1000 kg/m^3.
(13.08 MNm)

2. A vertical wall separates seawater on one side from fresh water on the other side. The seawater is 3.5 m deep and has a density of 1030 kg/m^3. The fresh water is 2 m deep and has a density of 1000 kg/m^3. Calculate the turning moment produced about the bottom for a unit width.
(59.12 kNm)

WORKED EXAMPLE No.5

A concrete wall retains water and has a hatch blocking off an outflow tunnel as shown. Find the total force on the hatch and the position of the centre of pressure. Calculate the total moment about the bottom edge of the hatch. The water density is1000 kg/m^3.

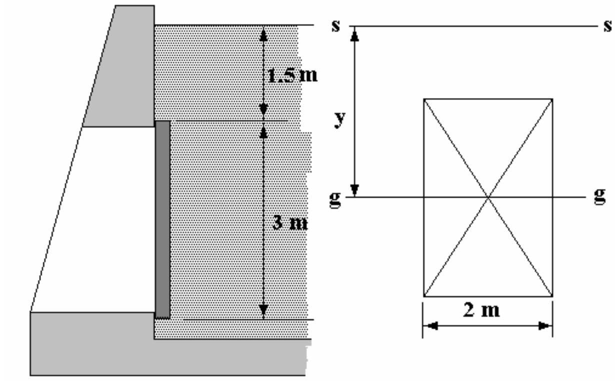


Fig.7

SOLUTION

Total force = R = ρ g A y
For the rectangle shown y = (1.5 + 3/2) = 3 m. A = 2 x 3 = 6 m^2.

R = 1000 x 9.81 x 6 x 3 = 176580 N or 176.58 kN

h = 2nd mom. of Area/ 1st mom. of Area

1st moment of Area = Ay = 6 x 3 = 18 m^3.

2nd mom of area = $\mathbf{I}_{SS}$ = (BD^3/12) + Ay^2= (2 x 3^3/12) + (6 x 3^2)

$\mathbf{I}_{SS}$= 4.5 + 54 = 58.5 m^4.

h = 58.5/18 = 3.25 m

The distance from the bottom edge is x = 4.5 – 3.25 = 1.25 m
Moment about the bottom edge is = Rx = 176.58 x 1.25 = ***220.725 kNm.***

WORKED EXAMPLE No.6

Find the force required at the top of the circular hatch shown in order to keep it closed against the water pressure outside. The density of the water is 1030 kg/m^3.

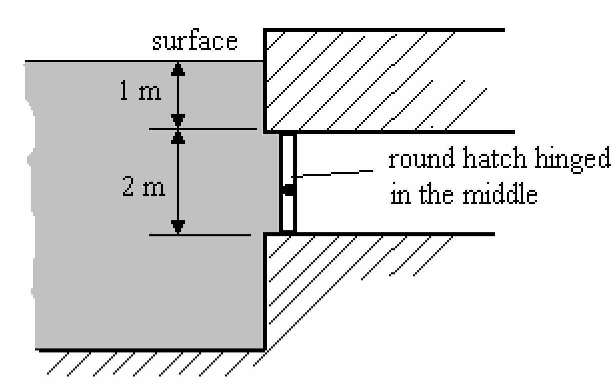


Fig. 8

y = 2 m from surface to middle of hatch.

Total Force = R = ρ g A y = 1030 x 9.81 x (π x 2^2/4) x 2 = 63487 N or 63.487 kN

Centre of Pressure h = 2nd moment/1st moment

2nd moment of area.

$\mathbf{I}_{SS} = \mathbf{I}_{gg} + Ay^2 = (\pi \times 2^4/64) + (\pi \times 2^2/4) \times 2^2$

$\mathbf{I}_{SS} = 13.3518\ m^4$.

1st moment of area

Ay = (π x 2^2/4) x 2 = 6.283 m^3.

Centre of pressure.

h = 13.3518/6.283 = 2.125 m

This is the depth at which, the total force may be assumed to act. Take moments about the hinge.

F = force at top.

R = force at centre of pressure which is 0.125 m below the hinge.

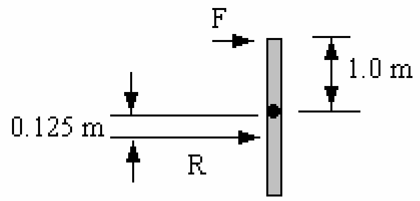


Fig. 9

For equilibrium F x 1 = 63.487 x 0.125

F = 7.936 kN

WORKED EXAMPLE No.7

The diagram shows a hinged circular vertical hatch diameter D that flips open when the water level outside reaches a critical depth h. Show that for this to happen the hinge must be located at a position x from the bottom given by the formula $x = \frac{D}{2}\left\{\frac{8h - 5D}{8h - 4D}\right\}$

Given that the hatch is 0.6 m diameter, calculate the position of the hinge such that the hatch flips open when the depth reaches 4 metres.

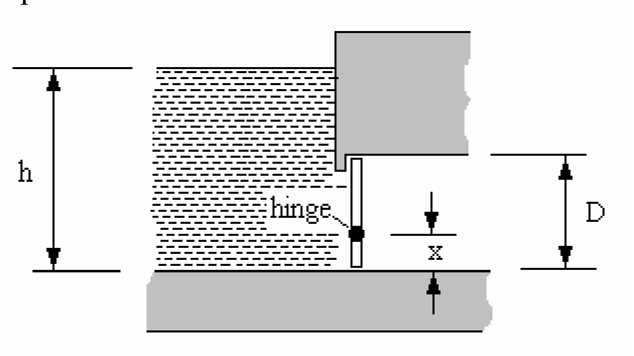


Fig.10

SOLUTION

The hatch will flip open as soon as the centre of pressure rises above the hinge creating a clockwise turning moment. When the centre of pressure is below the hinge, the turning moment is anticlockwise and the hatch is prevented from turning in that direction. We must make the centre of pressure at position x.

$$\bar{y} = h - \frac{D}{2}$$

$$\bar{h} = h - x$$

$$\bar{h} = \frac{\text{second moment of area}}{\text{first moment of area}} \text{ about the surface}$$

$$\bar{h} = \frac{I_{gg} + A\bar{y}^2}{A\bar{y}} = \frac{\frac{\pi D^4}{64} + \frac{\pi D^2}{4}\bar{y}^2}{\frac{\pi D^2}{4}\bar{y}} = \frac{D^2}{16\bar{y}} + \bar{y}$$

Equate for $\bar{h}$

$$\frac{D^2}{16\bar{y}} + \bar{y} = h - x$$

$$x = h - \frac{D^2}{16\bar{y}} - \bar{y} = h - \frac{D^2}{16\left(h - \frac{D}{2}\right)} - \left(h - \frac{D}{2}\right)$$

$$x = -\frac{D^2}{(16h - 8D)} + \frac{D}{2} = \frac{D}{2} - \frac{D^2}{(16h - 8D)}$$

$$x = \frac{D}{2}\left\{1 - \frac{D}{8h - 4D}\right\} = \frac{D}{2}\left\{\frac{8h - 4D - D}{8h - 4D}\right\} = \frac{D}{2}\left\{\frac{8h - 5D}{8h - 4D}\right\}$$

Putting D = 0.6 and h = 4 we get ***x = 0.5 m***

SELF ASSESSMENT EXERCISE No.3

1. A circular hatch is vertical and hinged at the bottom. It is 2 m diameter and the top edge is 2m below the free surface. Find the total force, the position of the centre of pressure and the force required at the top to keep it closed. The density of the water is 1000 kg/m^3.
(92.469 kN, 3.08 m,42.5 kN)

2. A large tank of sea water has a door in the side 1 m square. The top of the door is 5 m below the free surface. The door is hinged on the bottom edge. Calculate the force required at the top to keep it closed. The density of the sea water is 1036 kg/m^3.
(27.11 N)

3. A culvert in the side of a reservoir is closed by a vertical rectangular gate 2m wide and 1 m deep as shown in fig. 11. The gate is hinged about a horizontal axis which passes through the centre of the gate. The free surface of water in the reservoir is 2.5 m above the axis of the hinge. The density of water is 1000 kg/m^3.

   Assuming that the hinges are frictionless and that the culvert is open to atmosphere, determine

   (i) the force acting on the gate when closed due to the pressure of water. (55.897 kN)

   (ii) the moment to be applied about the hinge axis to open the gate. (1635 Nm)

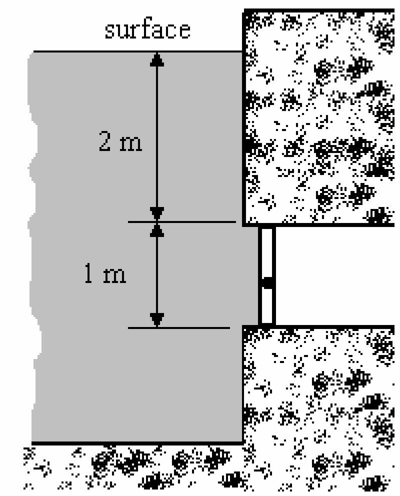


Fig.11

4. The diagram shows a rectangular vertical hatch breadth B and depth D. The hatch flips open when the water level outside reaches a critical depth h. Show that for this to happen the hinge must be located at a position x from the bottom given by the formula

$$x = \frac{D}{2}\left\{\frac{6h - 4D}{6h - 3D}\right\}$$

Given that the hatch is 1 m deep, calculate the position of the hinge such that the hatch flips open when the depth reaches 3 metres. (0.466 m)

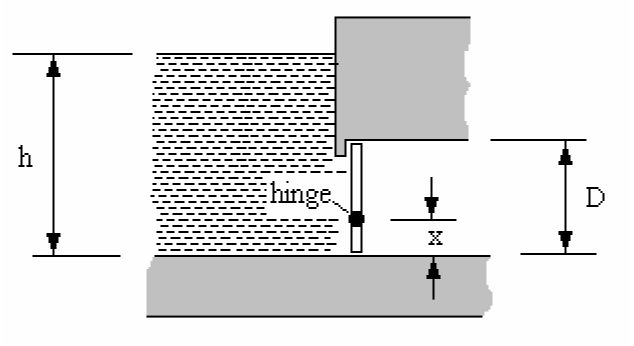


Fig.12

5. Fig.13 shows an L shaped spill gate that operates by pivoting about hinge O when the water level in the channel rises to a certain height H above O. A counterweight W attached to the gate provides closure of the gate at low water levels. With the channel empty the force at sill S is 1.635 kN. The distance l is 0.5m and the gate is 2 m wide.

Determine the magnitude of H.

(i) when the gate begins to open due to the hydrostatic load. (1 m)
(ii) when the force acting on the sill becomes a maximum. What is the magnitude of this force. (0.5 m)
Assume the effects of friction are negligible.

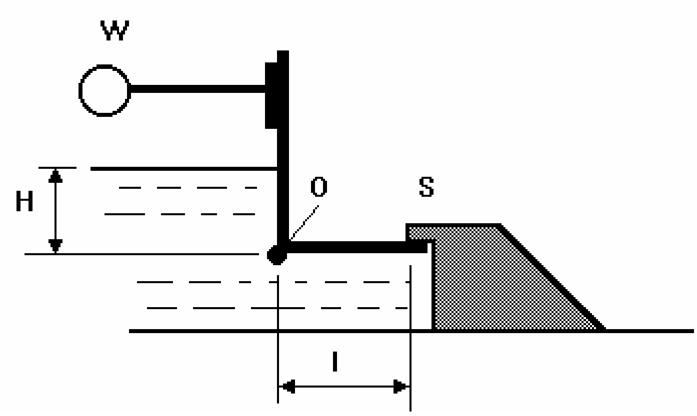


Fig.13

4 HYDROSTATIC DEVICES

In this section, you will study the following.

- Pascal's Laws.
- A simple hydraulic jack.
- Basic power hydraulic system.

4.1 PASCAL'S LAWS

- ***Pressure always acts normal to the surface of contact.***

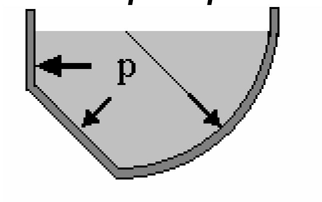


Fig.1.4

- ***The force of the molecules pushing on neighbouring molecules is equal in all directions so long as the fluid is static (still).***

- ***The force produced by a given pressure in a static fluid is the same on all equal areas.***

These statements are the basis of ***PASCAL'S LAWS*** and the unit of pressure is named after Pascal. These principles are used in simple devices giving force amplification.

4.2 CAR BRAKE

A simple hydraulic braking system is shown in fig.15

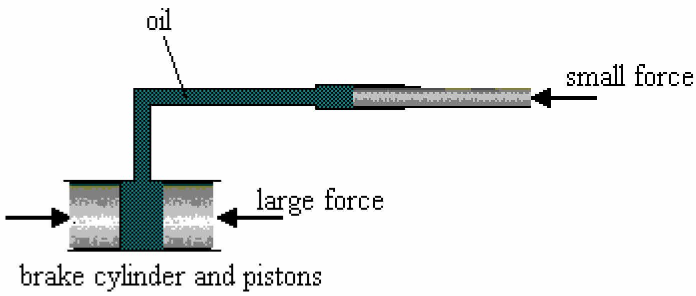


Fig.15

The small force produced by pushing the small piston produces pressure in the oil. The pressure acts on the larger pistons in the brake cylinder and produces a large force on the pistons that move the brake pads or shoes.

4.3 SIMPLE HYDRAULIC JACK

Fig. 16 shows the basis of a simple hydraulic jack.

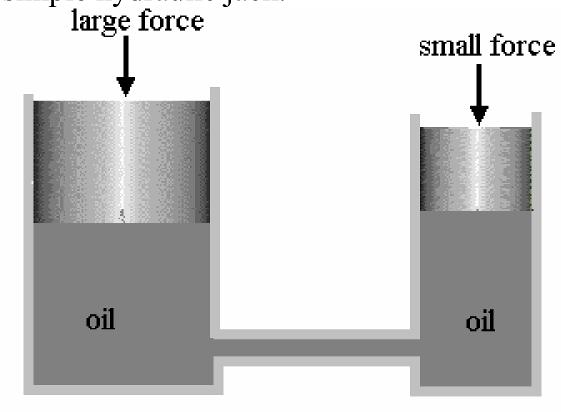


Fig.16

The small force pushing on the small piston produces a pressure in the oil. This pressure acts on the large piston and produces a larger force. This principle is used in most hydraulic systems but many modifications are needed to produce a really useful machine. In both the above examples, force is amplified because the same pressure acts on different piston areas. In order to calculate the force ratio we use the formula $p = F/A$.

FORCE RATIO

Let the small piston have an area A_1 and the large piston an area A_2. The force on the small piston is F_1 and on the large piston is F_2.

The pressure is the same for both pistons so $p = F_1/A_1 = F_2/A_2$

From this the force amplification ratio is $F_2/F_1 = A_2/A_1$

Note the area ratio is not the same as the diameter ratio. If the diameters are D_1 and D_2 then the ratio becomes $F_2/F_1 = A_2/A_1 = D_2^2/D_1^2$

MOVEMENT RATIO

The simple hydraulic jack produces force amplification but it is not possible to produce an increase in the energy, power or work. It follows that if no energy is lost nor gained, the large piston must move a smaller distance than the small piston.

Remember that work done is force x distance moved. $W = F\,x$

Let the small piston move a distance x_1 and the large piston x_2. The work input at the small piston is equal to the work out at the large piston so

$F_1x_1 = F_2x_2$ Substituting that $F_1 = pA_1$ and $F_2 = pA_2$

$pA_1\,x_1 = pA_2\,x_2$ or $A_1\,x_1 = A_2\,x_2$

The movement of the small piston as a ratio to the movement of the large piston is then $x_1/x2 = A_2/A_1$ = area ratio

4.4 PRACTICAL LIFTING JACK

A practical hydraulic jack uses a small pumping piston as shown. When this moves forward, the non-return valve NRV1 opens and NRV2 closes. Oil is pushed under the load piston and moves it up. When the piston moves back, NRV1 closes and NRV2 opens and replenishes the pumping cylinder from the reservoir. Successive operations of the pump raises the load piston. The oil release valve , when open, allows the oil under the load cylinder to return to the reservoir and lowers the load.

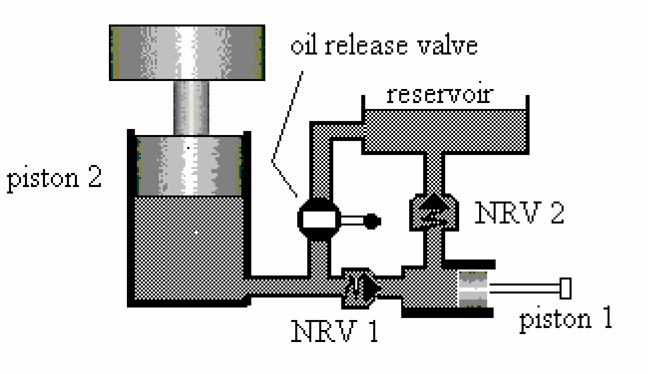


Fig.17

WORKED EXAMPLE No.8

A simple lifting jack has a pump piston 12 mm diameter and a load piston 60 mm diameter. Calculate the force needed on the pumping piston to raise a load of 8 kN. Calculate the pressure in the oil.

SOLUTION

Force Ratio = $A_2/A_1 = D_2^2/D_1^2 = (60/12)^2 = 25$

Force on the pumping piston is 1/25 of the load.

$F_1 = 8 \text{ x } 10^3/25 =$ ***320 N***

Pressure = Force/Area. Choosing the small piston
$A_1 = \pi D_1^2/4 = \pi \text{ x } 0.012^2/4 = 113.1 \text{ x } 10^{-6} \text{ m}^2$
$p = F/A = 320 / 113.1 \text{ x } 10^{-6} = 2.829 \text{ x } 10^6$ Pa or ***2.829 MPa***

Check using the large piston data.

$F_2 = 8 \text{ x } 10^3$ N
$A_2 = \pi D_2^2/4 = \pi \text{ x } 0.06^2/4 = 2.827 \text{ x } 10^{-3} \text{ m}^2$
$p = F/A = 8 \text{ x } 10^3 / 2.827 \text{ x } 10^{-3} = 2.829 \text{ x } 10^6$ Pa or ***2.829 MPa***

SELF ASSESSMENT EXERCISE No.4

1. Calculate F_1 and x_2 for the case shown below. (83.3 N, 555 mm)

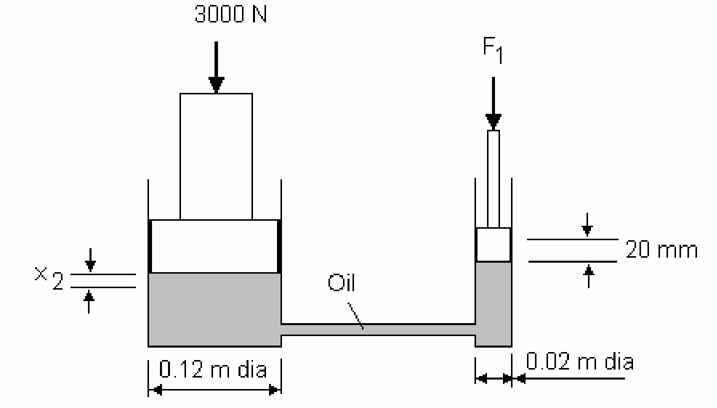


Fig.18

2. Calculate F_1 and x_1 for the case shown below. (312.5 kN, 6.25 mm)

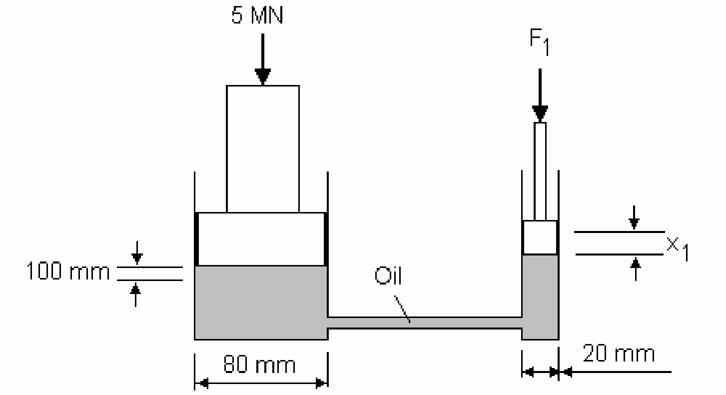


Fig.19

4.5 CYLINDERS

Cylinders are linear actuators that convert fluid power into mechanical power. They are also known as JACKS or RAMS. Hydraulic cylinders are used at high pressures. They produce large forces with precise movement. They are constructed of strong materials such as steel and designed to withstand large forces.

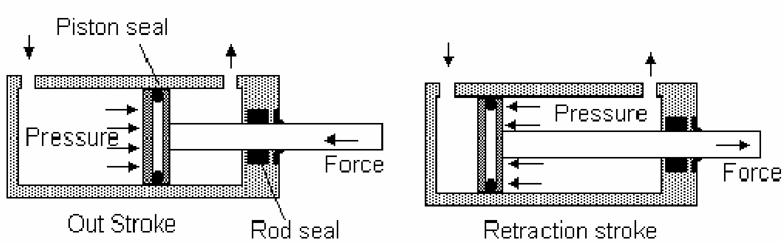


Fig.20

The diagram shows a double acting cylinder. Assume that the pressure on the other side of the piston is atmospheric. In this case, if we use gauge pressure, we need not worry about the atmospheric pressure. A is the full area of the piston. If the pressure is acting on the rod side, then the area is (A - a) where a is the area of the rod.

$F = pA$ on the full area of piston.

$F = p(A-a)$ on the rod side.

This force acting on the load is often less because of friction between the piston and piston rod and the seals.

WORKED EXAMPLE No.9

A single rod hydraulic cylinder must pull with a force of 5 kN. The piston is 75 mm diameter and the rod is 30 mm diameter. Calculate the pressure required.

SOLUTION

The pressure is required on the annular face of the piston in order to pull. The area acted on by the pressure is A - a

$A = \pi \times 0.075^2/4 = 4.418 \times 10^{-3}\ m^2$

$a = \pi \times 0.03^2/4 = 706.8 \times 10^{-6}\ m^2$

$A - a = 3.711 \times 10^{-3}\ m^2$

$p = F/(A - a) = 5 \times 10^3 / 3.711 \times 10^{-3} = 1.347 \times 10^6$ Pa or ***1.347 MPa***

4.6 BASIC HYDRAULIC POWER SYSTEM

The hand pump is replaced by a power driven pump. The load piston may be double acting so a directional valve is needed to direct the fluid from the pump to the top or bottom of the piston. The valve also allows the venting oil back to the reservoir.

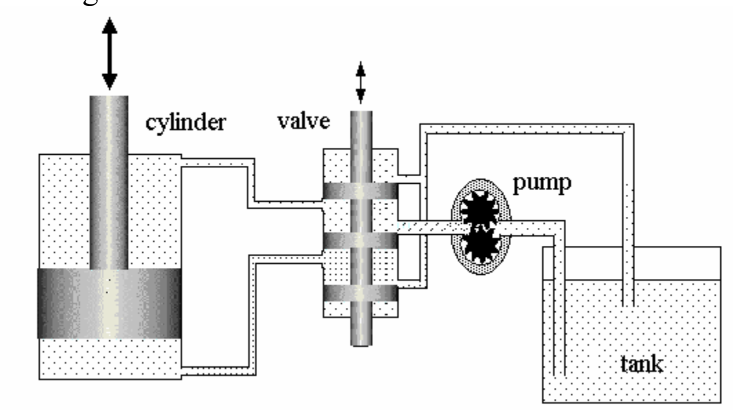


Fig.21

4.7 ACCUMULATORS

An accumulator is a device for storing pressurised liquid. One reason for this might be to act as an emergency power source when the pump fails.

Originally, accumulators were made of long hydraulic cylinders mounted vertically with a load bearing down on them. If the hydraulic system failed, the load pushed the piston down and expelled the stored liquid

Modern accumulators use high pressure gas (Nitrogen) and when the pump fails the gas expels the liquid.

WORKED EXAMPLE No.10

A simple accumulator is shown in fig.22. The piston is 200 mm diameter and the pressure of the liquid must be maintained at 30 MPa. Calculate the mass needed to produce this pressure.

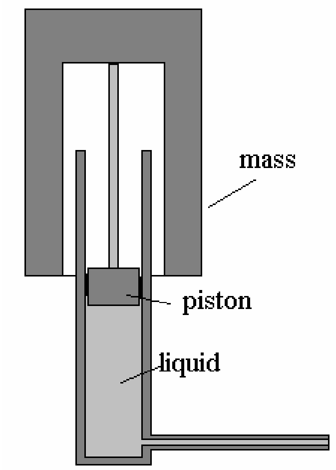


Fig.22

SOLUTION

Weight = pressure x area

Area = $\pi D^2/4 = \pi \times 0.2^2/4 = 0.0314\ m^2$

Weight = $30 \times 10^6 \times 0.0314 = 942.5 \times 10^3$ N or 942.5 kN

Mass = Weight/gravity = $942.5 \times 10^3 /9.81 = 96.073 \times 10^3$ kg or 96.073 Tonne

SELF ASSESSMENT EXERCISE No.5

1. A double acting hydraulic cylinder with a single rod must produce a thrust of 800 kN. The operating pressure is 100 bar gauge. Calculate the bore diameter required. (101.8 mm)

2. The cylinder in question 1 has a rod diameter of 25 mm. If the pressure is the same on the retraction (negative) stroke, what would be the force available? (795 kN)

3. A single acting hydraulic cylinder has a piston 75 mm diameter and is supplied with oil at 100 bar gauge. Calculate the thrust. (44.18 kN)

4. A vertical hydraulic cylinder (fig.22) is used to support a weight of 50 kN. The piston is 100 mm diameter. Calculate the pressure required. (6.37 MPa)

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FLUID MECHANICS H1 UNIT 8

NQF LEVEL 4

OUTCOME 2 VISCOSITY

TUTORIAL 2 – THE VISCOUS NATURE OF FLUIDS

2 Viscosity

Viscosity: shear stress; shear rate; dynamic viscosity; kinematic viscosity

Viscosity measurement: operating principles and limitations of viscosity measuring devices (eg falling sphere, capillary tube, rotational and orifice viscometers)

***Real fluids*:** Newtonian fluids; non-Newtonian fluids including pseudoplastic, Bingham plastic, Casson plastic and dilatent fluids

On completion of chapter 2 you should be able to do the following.

- Define viscosity and its units.
- Define a Newtonian fluid.
- Explain laminar flow.
- Explain turbulent flow.
- Explain fluid friction.
- Solve problems involving all the above.

Let's start by examining the meaning of viscosity.

1. VISCOSITY

1.1 BASIC THEORY

Molecules of fluids exert forces of attraction on each other. In liquids this is strong enough to keep the mass together but not strong enough to keep it rigid. In gases these forces are very weak and cannot hold the mass together.

When a fluid flows over a surface, the layer next to the surface may become attached to it (it wets the surface). The layers of fluid above the surface are moving so there must be shearing taking place between the layers of the fluid.

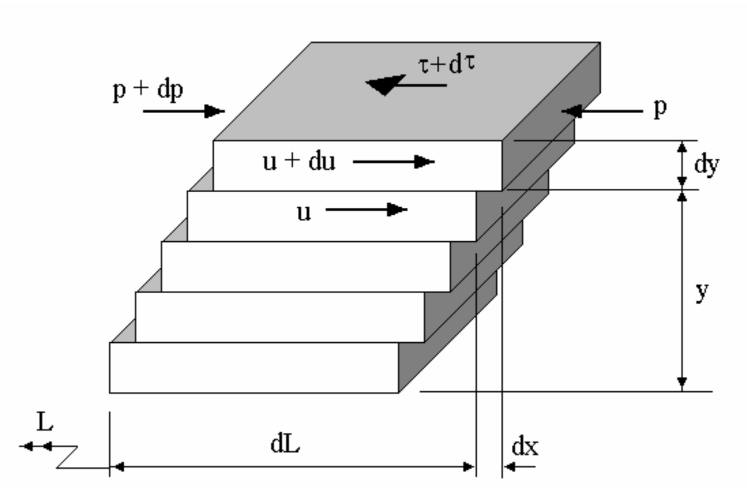


Fig.2.1

Let us suppose that the fluid is flowing over a flat surface in laminated layers from left to right as shown in figure 2.1.

y is the distance above the solid surface (no slip surface)
L is an arbitrary distance from a point upstream.
dy is the thickness of each layer.
dL is the length of the layer.
dx is the distance moved by each layer relative to the one below in a corresponding time dt.
u is the velocity of any layer.
du is the increase in velocity between two adjacent layers.

Each layer moves a distance dx in time dt relative to the layer below it. The ratio dx/dt must be the change in velocity between layers so du = dx/dt.

When any material is deformed sideways by a (shear) force acting in the same direction, a shear stress τ is produced between the layers and a corresponding shear strain γ is produced. Shear strain is defined as follows.

$$\gamma = \frac{\text{sideways deformation}}{\text{height of the layer being deformed}} = \frac{dx}{dy}$$

The rate of shear strain is defined as follows.

$$\dot{\gamma} = \frac{\text{shear strain}}{\text{time taken}} = \frac{\gamma}{dt} = \frac{dx}{dt\,dy} = \frac{du}{dy}$$

It is found that fluids such as water, oil and air, behave in such a manner that the shear stress between layers is directly proportional to the rate of shear strain.

$\tau = \text{constant x } \dot{\gamma}$

Fluids that obey this law are called ***NEWTONIAN FLUIDS***.

It is the constant in this formula that we know as the dynamic viscosity of the fluid.

$$\textbf{\textit{DYNAMIC VISCOSITY}}\ \mu = \frac{\text{shear stress}}{\text{rate of shear}} = \frac{\tau}{\dot{\gamma}} = \tau\frac{dy}{du}$$

FORCE BALANCE and VELOCITY DISTRIBUTION

A shear stress τ exists between each layer and this increases by $d\tau$ over each layer. The pressure difference between the downstream end and the upstream end is dp.

The pressure change is needed to overcome the shear stress. The total force on a layer must be zero so balancing forces on one layer (assumed 1 m wide) we get the following.

$$dp\,dy + d\tau\,dL = 0$$

$$\frac{d\tau}{dy} = -\frac{dp}{dL}$$

It is normally assumed that the pressure declines uniformly with distance downstream so the ***pressure gradient*** $\frac{dp}{dL}$ is assumed constant. The minus sign indicates that the pressure falls with distance. Integrating between the no slip surface (y = 0) and any height y we get

$$-\frac{dp}{dL} = \frac{d\tau}{dy} = \frac{d\left(\mu\frac{du}{dy}\right)}{dy}$$

$$-\frac{dp}{dL} = \mu\frac{d^2u}{dy^2} \quad \text{..........} \quad \text{....(2.1)}$$

Integrating twice to solve u we get the following.

$$-y\frac{dp}{dL} = \mu\frac{du}{dy} + A$$

$$-\frac{y^2}{2}\frac{dp}{dL} = \mu u + Ay + B$$

A and B are constants of integration that should be solved based on the known conditions (boundary conditions). For the flat surface considered in figure 2.1 one boundary condition is that u = 0 when y = 0 (the no slip surface). Substitution reveals the following.

0 = 0 +0 +B   hence B = 0

At some height δ above the surface, the velocity will reach the mainstream velocity u_o. This gives us the second boundary condition $u = u_o$ when $y = \delta$. Substituting we find the following.

$$-\frac{\delta^2}{2}\frac{dp}{dL} = \mu u_o + A\delta$$

$$A = -\frac{\delta}{2}\frac{dp}{dL} - \frac{\mu u_o}{\delta} \quad \text{hence}$$

$$-\frac{y^2}{2}\frac{dp}{dL} = \mu u + \left(-\frac{\delta}{2}\frac{dp}{dL} - \frac{\mu u_o}{\delta}\right)y$$

$$u = y\left(\frac{\delta}{2\mu}\frac{dp}{dL} + \frac{u_o}{\delta}\right)$$

Plotting u against y gives figure 2.2.

BOUNDARY LAYER.

The velocity grows from zero at the surface to a maximum at height δ. In theory, the value of δ is infinity but in practice it is taken as the height needed to obtain 99% of the mainstream velocity. This layer is called the boundary layer and δ is the boundary layer thickness. It is a very important concept and is discussed more fully in chapter 3. The inverse gradient of the boundary layer is du/dy and this is the rate of shear strain γ.

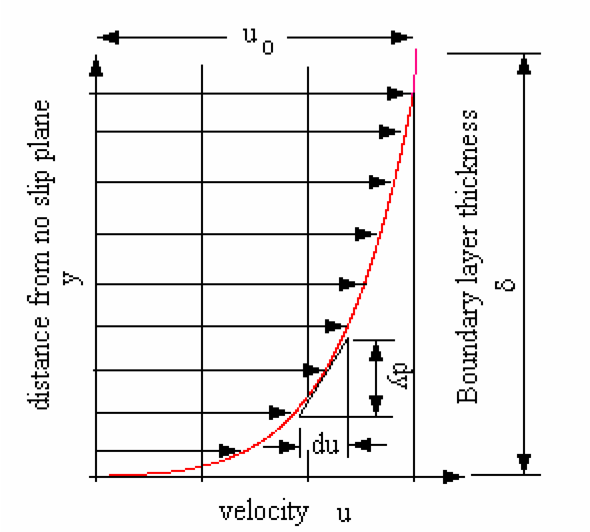


Fig.2.2

1.2. UNITS of VISCOSITY

1.2.1 DYNAMIC VISCOSITY μ

The units of dynamic viscosity μ are N s/m^2. It is normal in the international system (SI) to give a name to a compound unit. The old metric unit was a dyne.s/cm^2 and this was called a POISE after Poiseuille. It follows that the SI unit is related to the Poise such that 10 Poise = 1 Ns/m^2
This is not an acceptable multiple. Since, however, 1 CentiPoise (1cP) is 0.001 N s/m^2 then the cP is the accepted SI unit.

1cP = 0.001 N s/m^2.

The symbol η is also commonly used for dynamic viscosity.

There are other ways of expressing viscosity and this is covered next.

1.2.2 KINEMATIC VISCOSITY ν

This is defined as follows. $\nu = \dfrac{\text{dynamic viscosity}}{\text{density}} = \dfrac{\mu}{\rho}$

The basic units are m^2/s. The old metric unit was the cm^2/s and this was called the STOKE after the British scientist. It follows that 1 Stoke (St) = 0.0001 m^2/s and this is not an acceptable SI multiple. The centiStoke (cSt) ,however, is 0.000001 m^2/s and this is an acceptable multiple.

$$1cSt = 0.000001\ m^2/s = 1\ mm^2/s$$

1.2.3 OTHER UNITS

Other units of viscosity have come about because of the way viscosity is measured. For example REDWOOD SECONDS comes from the name of the Redwood viscometer. Other units are Engler Degrees, SAE numbers and so on. Conversion charts and formulae are available to convert them into useable engineering or SI units.

1.2.4 VISCOMETERS

The measurement of viscosity is a large and complicated subject. The principles rely on the resistance to flow or the resistance to motion through a fluid. Many of these are covered in British Standards 188. The following is a brief description of some types.

U TUBE VISCOMETER

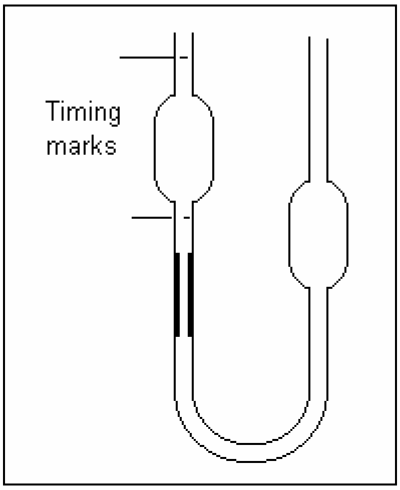


Fig.2.3

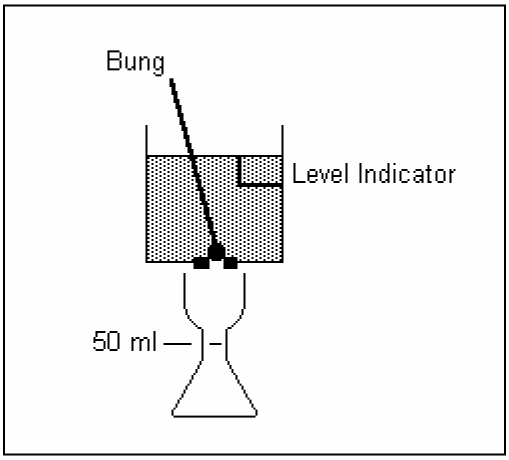
The fluid is drawn up into a reservoir and allowed to run through a capillary tube to another reservoir in the other limb of the U tube.

The time taken for the level to fall between the marks is converted into cSt by multiplying the time by the viscometer constant.

$$\nu = ct$$

The constant c should be accurately obtained by calibrating the viscometer against a master viscometer from a standards laboratory.

REDWOOD VISCOMETER

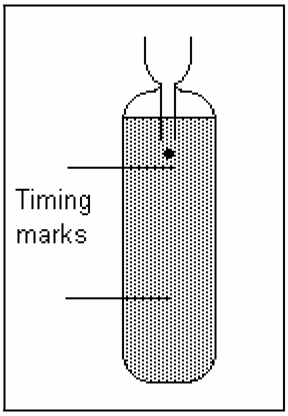


This works on the principle of allowing the fluid to run through an orifice of very accurate size in an agate block.

50 ml of fluid are allowed to empty from the level indicator into a measuring flask. The time taken is the viscosity in Redwood seconds. There are two sizes giving Redwood No.1 or No.2 seconds. These units are converted into engineering units with tables.

Fig.2.4

FALLING SPHERE VISCOMETER



This viscometer is covered in BS188 and is based on measuring the time for a small sphere to fall in a viscous fluid from one level to another. The buoyant weight of the sphere is balanced by the fluid resistance and the sphere falls with a constant velocity. The theory is based on Stoke's Law and is only valid for very slow velocities. The theory is covered later in the section on laminar flow where it is shown that the terminal velocity (u) of the sphere is related to the dynamic viscosity (μ) and the density of the fluid and sphere (ρ_f and ρ_S) by the formula

$$\mu = F\ gd^2(\rho_S - \rho_f)/18u$$

Fig.2.5

F is a correction factor called the Faxen correction factor, which takes into account a reduction in the velocity due to the effect of the fluid being constrained to flow between the wall of the tube and the sphere.

ROTATIONAL TYPES

There are many types of viscometers, which use the principle that it requires a torque to rotate or oscillate a disc or cylinder in a fluid. The torque is related to the viscosity. Modern instruments consist of a small electric motor, which spins a disc or cylinder in the fluid. The torsion of the connecting shaft is measured and processed into a digital readout of the viscosity in engineering units.

You should now find out more details about viscometers by reading BS188, suitable textbooks or literature from oil companies.

SELF ASSESSMENT EXERCISE No. No. 1

1. Describe the principle of operation of the following types of viscometers.

   a. Redwood Viscometers.

   b. British Standard 188 glass U tube viscometer.

   c. British Standard 188 Falling Sphere Viscometer.

   d. Any form of Rotational Viscometer

2. NON-NEWTONIAN FLUIDS

Consider figure 2.6. This shows the relationship between shear stress τ and rate of shear strain γ.

Graph A shows an ideal fluid that has no viscosity and hence has no shear stress at any point. This is often used in theoretical models of fluid flow.

Graph B shows a Newtonian Fluid. This is the type of fluid with which this book is mostly concerned, fluids such as water and oil. A Newtonian fluid obeys the rule $\tau = \mu\ du/dy = \mu\ \gamma$. The graph is hence a straight line and the gradient is the viscosity μ.

There is a range of other liquid or semi-liquid materials that do not obey this law and produce strange flow characteristics. Such materials include various foodstuffs, paints, cements and so on. Many of these are in fact solid particles suspended in a liquid with various concentrations.

Graph C shows the relationship for a ***Dilatent fluid***. The gradient and hence viscosity increases with γ and such fluids are also called ***shear-thickening.*** This phenomenon occurs with some solutions of sugar and starches.

Graph D shows the relationship for a ***Pseudo-plastic***. The gradient and hence viscosity reduces with γ and they are called ***shear-thinning.*** Most foodstuffs are like this as well as clay and liquid cement..

Other fluids behave like a ***plastic*** and require a minimum stress before it shears τ_y. This is plastic behaviour but unlike plastics, there may be no elasticity prior to shearing.

Graph E shows the relationship for a ***Bingham plastic.*** This is the special case where the behaviour is the same as a Newtonian fluid except for the existence of the yield stress. Foodstuffs containing high level of fats approximate to this model (butter, margarine, chocolate and Mayonnaise).

Graph F shows the relationship for a ***plastic*** fluid that exhibits shear thickening characteristics.

Graph G shows the relationship for a ***Casson fluid***. This is a plastic fluid that exhibits shear-thinning characteristics. This model was developed for fluids containing rod like solids and is often applied to molten chocolate and blood.

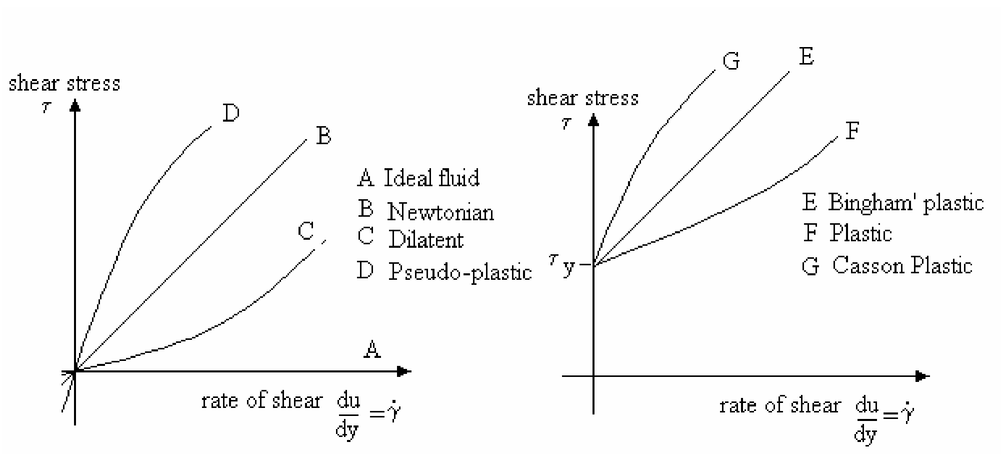


Fig.2.6

MATHEMATICAL MODELS

The graphs that relate shear stress τ and rate of shear strain γ are based on models or equations. Most are mathematical equations created to represent empirical data.

Hirschel and Bulkeley developed the power law for non-Newtonian equations.This is as follows.

$\tau = \tau_y + K\dot{\gamma}^n$ K is called the consistency coefficient and n is a power.

In the case of a Newtonian fluid n = 1 and $\tau_y = 0$ and K = μ (the dynamic viscosity) $\tau = \mu\dot{\gamma}$

For a Bingham plastic, n = 1 and K is also called the plastic viscosity μ_p. The relationship reduces to $\tau = \tau_y + \mu_p\dot{\gamma}$

For a ***dilatent fluid,*** $\tau_y = 0$ and n>1

For a ***pseudo-plastic,*** $\tau_y = 0$ and n<1

The model for both is $\tau = K\dot{\gamma}^n$

The ***Herchel-Bulkeley*** model is as follows. $\tau = \tau_y + K\dot{\gamma}^n$

This may be developed as follows.

$\tau = \tau_y + K\dot{\gamma}^n$

$\tau - \tau_y = K\dot{\gamma}^n$ sometimes writtend as $\tau - \tau_y = \mu_p\dot{\gamma}^n$ where μ_p is called the plastic viscosity.

dividing by $\dot{\gamma}$

$$\frac{\tau}{\dot{\gamma}} - \frac{\tau_y}{\dot{\gamma}} = K\frac{\dot{\gamma}^n}{\dot{\gamma}} = K\dot{\gamma}^{n-1}$$

$$\frac{\tau}{\dot{\gamma}} = \frac{\tau_y}{\dot{\gamma}} + K\dot{\gamma}^{n-1}$$ The ratio is called the apparent viscosity μ_{app}

$$\mu_{app} = \frac{\tau}{\dot{\gamma}} = \frac{\tau_y}{\dot{\gamma}} + K\dot{\gamma}^{n-1}$$

For a Bingham plastic $n = 1$ so $\mu_{app} = \frac{\tau_y}{\dot{\gamma}} + K$

For a Fluid with no yield shear value $\tau_y = 0$ so $\mu_{app} = K\dot{\gamma}^{n-1}$

The ***Casson*** fluid model is quite different in form from the others and is as follows.

$$\tau^{\frac{1}{2}} = \tau_y^{\frac{1}{2}} + K\dot{\gamma}^{\frac{1}{2}}$$

Note that fluids with a shear yield stress will flow in a pipe as a plug. Within a certain radius, the shear stress will be insufficient to produce shearing so inside that radius the fluid flows as a solid plug.

WORKED EXAMPLE No. 1

The Herchel-Bulkeley model for a non-Newtonian fluid is as follows. $\tau = \tau_y + K\dot{\gamma}^n$.

Derive an equation for the minimum pressure required drop per metre length in a straight horizontal pipe that will produce flow.

Given that the pressure drop per metre length in the pipe is 60 Pa/m and the yield shear stress is 0.2 Pa, calculate the radius of the slug sliding through the middle.

SOLUTION

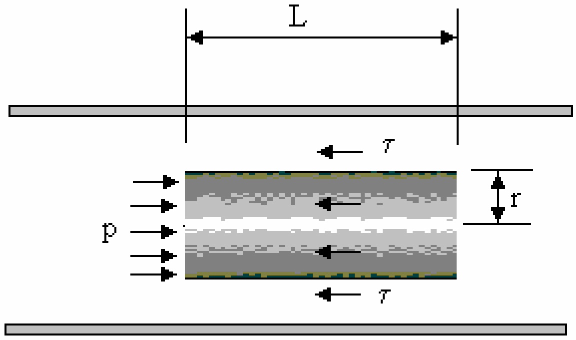


Fig. 2.7

The pressure difference p acting on the cross sectional area must produce sufficient force to overcome the shear stress τ acting on the surface area of the cylindrical slug. For the slug to move, the shear stress must be at least equal to the yield value τy. Balancing the forces gives the following.

$p \times \pi r^2 = \tau_y \times 2\pi rL$

$p/L = 2\tau_y/r$

$60 = 2 \times 0.2/r \qquad r = 0.4/60 = 0.0066$ m or 6.6 mm

WORKED EXAMPLE No. 2

A Bingham plastic flows in a pipe and it is observed that the central plug is 30 mm diameter when the pressure drop is 100 Pa/m.

Calculate the yield shear stress.

Given that at a larger radius the rate of shear strain is 20 s^{-1} and the consistency coefficient is 0.6 Pa s, calculate the shear stress.

SOLUTION

For a Bingham plastic, the same theory as in the last example applies.

$p/L = 2\tau_y/r \qquad 100 = 2\,\tau_y/0.015 \qquad \tau_y = 100 \times 0.015/2 = 0.75$ Pa

A mathematical model for a Bingham plastic is

$\tau = \tau_y + K\dot{\gamma} = 0.75 + 0.6 \times 20 = 12.75$ Pa

WORKED EXAMPLE No. 3

Research has shown that tomato ketchup has the following viscous properties at 25°C.

Consistency coefficient K = 18.7 Pa s^n
Power n = 0.27
Shear yield stress = 32 Pa

Calculate the apparent viscosity when the rate of shear is 1, 10, 100 and 1000 s^{-1} and conclude on the effect of the shear rate on the apparent viscosity.

SOLUTION

This fluid should obey the Herchel-Bulkeley equation so

$$\mu_{app} = \frac{\tau_y}{\dot{\gamma}} + K\dot{\gamma}^{n-1}$$

$$\mu_{app} = \frac{32}{\dot{\gamma}} + 18.7\dot{\gamma}^{0.27-1}$$

Evaluating at the various strain rates we get.

$\gamma = 1$ $\mu_{app} = 18.8$
$\gamma = 10$ $\mu_{app} = 3.482$
$\gamma = 100$ $\mu_{app} = 0.648$
$\gamma = 1000$ $\mu_{app} = 0.12$

The apparent viscosity reduces as the shear rate increases.

SELF ASSESSMENT EXERCISE No. 2

Find examples of the following non- Newtonian fluids by searching the web.

Pseudo Plastic

Bingham's Plastic

Casson Plastic

Dilatent Fluid

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NQF LEVEL 4

OUTCOME 3 - THE FLOW OF REAL FLUIDS

TUTORIAL 3

3 **Flow of real fluids**

Head losses**:** head loss in pipes by Darcy's formula; Moody diagram; head loss due to sudden enlargement and contraction of pipe diameter; head loss at entrance to a pipe; head loss in valves; flow between reservoirs due to gravity; hydraulic gradient; siphons; hammer blow in pipes

Reynolds' number**:** inertia and viscous resistance forces; laminar and turbulent flow; critical velocities

Viscous drag**:** dynamic pressure; form drag; skin friction drag; drag coefficient

Dimensional analysis**:** checking validity of equations such as those for pressure at depth; thrust on immersed surfaces and impact of a jet; forecasting the form of possible equations such as those for Darcy's formula and critical velocity in pipes

On completion of this outcome you should be able to do the following.

- Derive Bernoulli's equation for liquids.
- Define and explain laminar and turbulent flow.
- Find the pressure losses in piped systems due to fluid friction.
- Find the minor frictional losses in piped systems.
- Describe and calculate the effect of hammer blow in pipes.
- Derive the basic relationship between pressure, velocity and force.

This is a very large outcome requiring a lot of study time. The tutorial may contain more material than needed by those who have already studied the appropriate pre-requisite material.

Let's start by revising basics. The flow of a fluid in a pipe depends upon two fundamental laws, the conservation of mass and energy.

3.1 PIPE FLOW

The solution of pipe flow problems requires the applications of two principles, the law of conservation of mass (continuity equation) and the law of conservation of energy (Bernoulli's equation)

3.1.1 CONSERVATION OF MASS

When a fluid flows at a constant rate in a pipe or duct, the mass flow rate must be the same at all points along the length. Consider a liquid being pumped into a tank as shown (fig.3.1).

The mass flow rate at any section is $m = \rho A u_m$

ρ = density (kg/m^3) u_m = mean velocity (m/s) A = Cross Sectional Area (m^2)

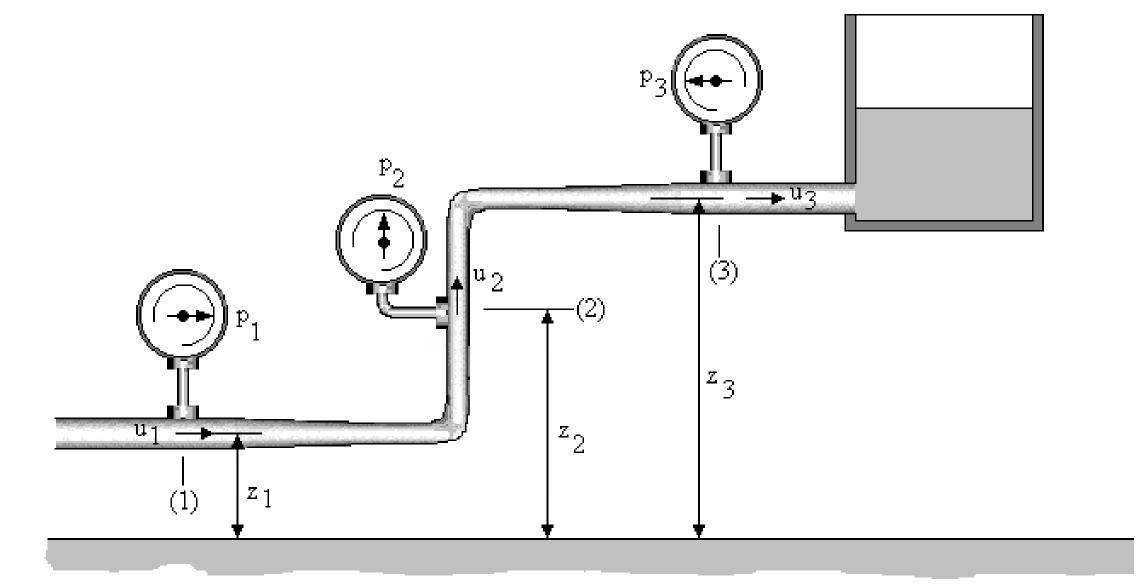


Fig.3.1

For the system shown the mass flow rate at (1), (2) and (3) must be the same so

$$\rho_1 A_1 u_1 = \rho_2 A_2 u_2 = \rho_3 A_3 u_3$$

In the case of liquids the density is equal and cancels so

$$A_1 u_1 = A_2 u_2 = A_3 u_3 = Q$$

3.1.2 CONSERVATION OF ENERGY

ENERGY FORMS

FLOW ENERGY
This is the energy a fluid possesses by virtue of its pressure.
The formula is ***F.E. = pQ Joules***

p is the pressure (Pascals)
Q is volume rate (m^3)

POTENTIAL OR GRAVITATIONAL ENERGY

This is the energy a fluid possesses by virtue of its altitude relative to a datum level.

The formula is ***P.E. = mgz Joules***

m is mass (kg)
z is altitude (m)

KINETIC ENERGY

This is the energy a fluid possesses by virtue of its velocity.

The formula is $\boldsymbol{K.E. = \frac{1}{2} m u_m^2}$ ***Joules***

u_m is mean velocity (m/s)

INTERNAL ENERGY

This is the energy a fluid possesses by virtue of its temperature. It is usually expressed relative to $0^{o}C$.

The formula is $\boldsymbol{U = mc\theta}$

c is the specific heat capacity (J/kg ^{o}C)
θ is the temperature in ^{o}C

In the following work, internal energy is not considered in the energy balance.

SPECIFIC ENERGY

Specific energy is the energy per kg so the three energy forms as specific energy are as follows.

$\boldsymbol{F.E./m = pQ/m = p/\rho}$ ***Joules/kg***
$\boldsymbol{P.E/m. = gz}$ ***Joules/kg***
$\boldsymbol{K.E./m = \frac{1}{2} u^2}$ ***Joules/kg***

ENERGY HEAD

If the energy terms are divided by the weight mg, the result is energy per Newton. Examining the units closely we have J/N = N m/N = metres.

It is normal to refer to the energy in this form as the energy head. The three energy terms expressed this way are as follows.

$\boldsymbol{F.E./mg = p/\rho g = h}$
$\boldsymbol{P.E./mg = z}$
$\boldsymbol{K.E./mg = u^2/2g}$

The flow energy term is called the pressure head and this follows since earlier it was shown $p/\rho g = h$. This is the height that the liquid would rise to in a vertical pipe connected to the system.

The potential energy term is the actual altitude relative to a datum.

The term $u^2/2g$ is called the kinetic head and this is the pressure head that would result if the velocity is converted into pressure.

3.1.3 BERNOULLI'S EQUATION

Bernoulli's equation is based on the conservation of energy. If no energy is added to the system as work or heat then the total energy of the fluid is conserved. Remember that internal (thermal energy) has not been included.

The total energy E_T at (1) and (2) on the diagram (fig.3.1) must be equal so :

$$E_T = p_1 Q_1 + mgz_1 + m\frac{u_1^2}{2} = p_2 Q_2 + mgz_2 + m\frac{u_2^2}{2}$$

Dividing by mass gives the specific energy form

$$\frac{E_T}{m} = \frac{p_1}{\rho_1} + gz_1 + \frac{u_1^2}{2} = \frac{p_2}{\rho_2} + gz_2 + \frac{u_2^2}{2}$$

Dividing by g gives the energy terms per unit weight

$$\frac{E_T}{mg} = \frac{p_1}{g\rho_1} + z_1 + \frac{u_1^2}{2g} = \frac{p_2}{g\rho_2} + z_2 + \frac{u_2^2}{2g}$$

Since $p/\rho g$ = pressure head h then the total head is given by the following.

$$h_T = h_1 + z_1 + \frac{u_1^2}{2g} = h_2 + z_2 + \frac{u_2^2}{2g}$$

This is the head form of the equation in which each term is an energy head in metres. z is the potential or gravitational head and $u^2/2g$ is the kinetic or velocity head.

For liquids the density is the same at both points so multiplying by ρg gives the pressure form. The total pressure is as follows.

$$p_T = p_1 + \rho g z_1 + \frac{\rho u_1^2}{2} = p_2 + \rho g z_2 + \frac{\rho u_2^2}{2}$$

In real systems there is friction in the pipe and elsewhere. This produces heat that is absorbed by the liquid causing a rise in the internal energy and hence the temperature. In fact the temperature rise will be very small except in extreme cases because it takes a lot of energy to raise the temperature. If the pipe is long, the energy might be lost as heat transfer to the surroundings. Since the equations did not include internal energy, the balance is lost and we need to add an extra term to the right side of the equation to maintain the balance. This term is either the head lost to friction h_L or the pressure loss p_L.

$$h_1 + z_1 + \frac{u_1^2}{2g} = h_2 + z_2 + \frac{u_2^2}{2g} + h_L$$

The pressure form of the equation is as follows.

$$p_1 + \rho g z_1 + \frac{\rho u_1^2}{2} = p_2 + \rho g z_2 + \frac{\rho u_2^2}{2} + p_L$$

The total energy of the fluid (excluding internal energy) is no longer constant.

Note that if one of the points is a ***free surface*** the pressure is normally atmospheric but if gauge pressures are used, the pressure and pressure head becomes zero. Also, if the surface area is large (say a large tank), the velocity of the surface is small and when squared becomes negligible so the kinetic energy term is neglected (made zero).

WORKED EXAMPLE No. 3.1

The diagram shows a pump delivering water through as pipe 30 mm bore to a tank. Find the pressure at point (1) when the flow rate is 1.4 dm^3/s. The density of water is 1000 kg/m^3. The loss of pressure due to friction is 50 kPa.

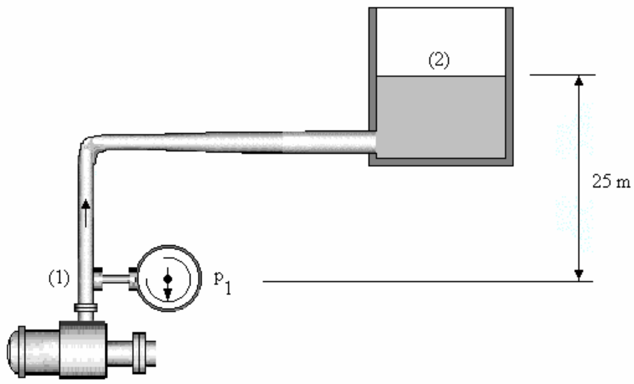


Fig.3.2

SOLUTION

Area of bore $A = \pi \times 0.03^2/4 = 706.8 \times 10^{-6}\ m^2$.
Flow rate $Q = 1.4\ dm^3/s = 0.0014\ m^3/s$
Mean velocity in pipe = Q/A = 1.98 m/s
Apply Bernoulli between point (1) and the surface of the tank.

$$p_1 + \rho g z_1 + \frac{\rho u_1^2}{2} = p_2 + \rho g z_2 + \frac{\rho u_2^2}{2} + p_L$$

Make the low level the datum level and $z_1 = 0$ and $z_2 = 25$.
The pressure on the surface is zero gauge pressure. $P_L = 50\ 000$ Pa
The velocity at (1) is 1.98 m/s and at the surface it is zero.

$$p_1 + 0 + \frac{1000\text{x}1.98^2}{2} = 0 + 1000\text{x}9.9125 + 0 + 50000 \qquad p_1 = 293.29\text{kPa gauge pressure}$$

WORKED EXAMPLE 3.2

The diagram shows a tank that is drained by a horizontal pipe. Calculate the pressure head at point (2) when the valve is partly closed so that the flow rate is reduced to 20 dm^3/s. The pressure loss is equal to 2 m head.

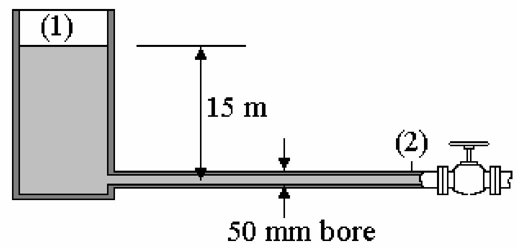


Fig.3.3

SOLUTION

Since point (1) is a free surface, $h_1 = 0$ and u_1 is assumed negligible.
The datum level is point (2) so $z_1 = 15$ and $z_2 = 0$.
Q = 0.02 m3/s
$A_2 = \pi d^2/4 = \pi \times (0.05^2)/4 = 1.963 \times 10^{-3}\ m^2$.
$u_2 = Q/A = 0.02/1.963 \times 10^{-3} = 10.18$ m/s

Bernoulli's equation in head form is as follows.

$$h_1 + z_1 + \frac{u_1^2}{2g} = h_2 + z_2 + \frac{u_2^2}{2g} + h_L$$

$$0 + 15 + 0 = h_2 + 0 + \frac{10.18^2}{2 \text{ x } 9.81} + 2$$

$$h_2 = 7.72\text{m}$$

WORKED EXAMPLE 3.3

The diagram shows a horizontal nozzle discharging into the atmosphere. The inlet has a bore area of 600 mm^2 and the exit has a bore area of 200 mm^2. Calculate the flow rate when the inlet pressure is 400 Pa. Assume there is no energy loss.

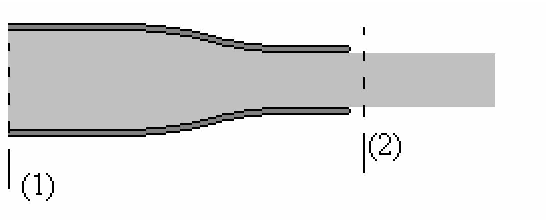


Fig. 3.4

SOLUTION

Apply Bernoulli between (1) and (2)

$$p_1 + \rho g z_1 + \frac{\rho u_1^2}{2} = p_2 + \rho g z_2 + \frac{\rho u_2^2}{2} + p_L$$

Using gauge pressure, p2 = 0 and being horizontal the potential terms cancel. The loss term is zero so the equation simplifies to the following.

$$p_1 + \frac{\rho u_1^2}{2} = \frac{\rho u_2^2}{2}$$

From the continuity equation we have

$$u_1 = \frac{Q}{A_1} = \frac{Q}{600 \text{ x } 10^{-6}} = 1666.7Q$$

$$u_2 = \frac{Q}{A_2} = \frac{Q}{200 \text{ x } 10^{-6}} = 5\,000\,Q$$

Putting this into Bernoulli's equation we have the following.

$$400 + 1000 \text{ x } \frac{(1666.7Q)^2}{2} = 1000 \text{ x } \frac{(5000Q)^2}{2}$$

$$400 + 1.389 \text{ x} 10^9 Q^2 = 12.5 \text{ x } 10^9 Q^2$$

$$400 = 11.11 \text{ x} 10^9 Q^2$$

$$Q^2 = \frac{400}{11.11 \text{ x} 10^9} = 36 \text{ x} 10^{-9}$$

$$Q = 189.7 \text{ x } 10^{-6} \text{ m}^3/\text{s or } 189.7 \text{ cm}^3/\text{s}$$

3.1.4 HYDRAULIC GRADIENT

Consider a tank draining into another tank at a lower level as shown. There are small vertical tubes at points along the length to indicate the pressure head (h). Relative to a datum, the total energy head is

$h_T = h + z + u^2/2g$

This is shown as line A.

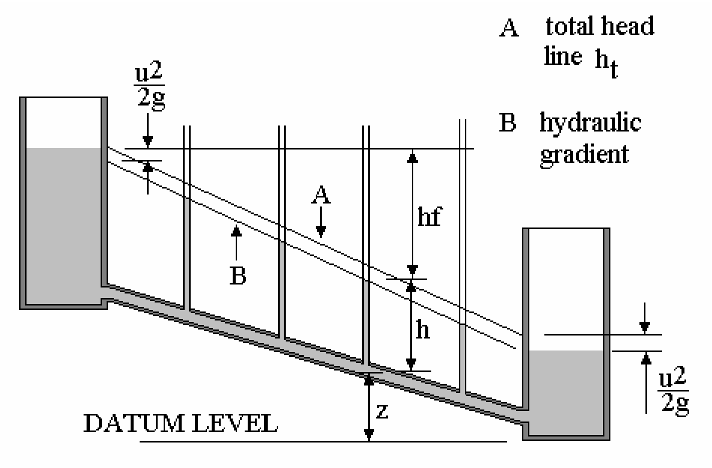


Fig.3.5

The hydraulic grade line is the line joining the free surfaces in the tubes and represents the sum of h and z only. This is shown as line B and it is always below the line of h_T by the velocity head $u^2/2g$. Note that at exit from the pipe, the velocity head is not recovered but lost as friction as the emerging jet collides with the static liquid. The free surface of the tank does not rise.

The only reason why the hydraulic grade line is not horizontal is because there is a frictional loss h_f. The actual gradient of the line at any point is the rate of change with length $i = \delta h_f/\delta L$

<u>SELF ASSESSMENT EXERCISE 3.1</u>

1. A pipe 100 mm bore diameter carries oil of density 900 kg/m^3 at a rate of 4 kg/s. The pipe reduces to 60 mm bore diameter and rises 120 m in altitude. The pressure at this point is atmospheric (zero gauge). Assuming no frictional losses, determine:

   i. The volume/s (4.44 dm^3/s)
   ii. The velocity at each section (0.566 m/s and 1.57 m/s)
   iii. The pressure at the lower end. (1.06 MPa)

2. A pipe 120 mm bore diameter carries water with a head of 3 m. The pipe descends 12 m in altitude and reduces to 80 mm bore diameter. The pressure head at this point is 13 m. The density is 1000 kg/m^3. Assuming no losses, determine

   i. The velocity in the small pipe (7 m/s)
   ii. The volume flow rate. (35 dm^3/s)

3. A horizontal nozzle reduces from 100 mm bore diameter at inlet to 50 mm at exit. It carries liquid of density 1000 kg/m^3 at a rate of 0.05 m^3/s. The pressure at the wide end is 500 kPa (gauge). Calculate the pressure at the narrow end neglecting friction. (196 kPa)

4. A pipe carries oil of density 800 kg/m^3. At a given point (1) the pipe has a bore area of 0.005 m^2 and the oil flows with a mean velocity of 4 m/s with a gauge pressure of 800 kPa. Point (2) is further along the pipe and there the bore area is 0.002 m^2 and the level is 50 m above point (1). Calculate the pressure at this point (2). Neglect friction. (374 kPa)

5. A horizontal nozzle has an inlet velocity u_1 and an outlet velocity u_2 and discharges into the atmosphere. Show that the velocity at exit is given by the following formulae.

   $u_2 = \{2\Delta p/\rho + u_1^2\}^{1/2}$

   and $u_2 = \{2g\Delta h + u_1^2\}^{1/2}$

3.2 LAMINAR and TURBULENT FLOW

The following work only applies to Newtonian fluids (chapter 2).

3.2.1 LAMINAR FLOW

A ***stream line*** is an imaginary line with no flow normal to it, only along it. When the flow is laminar, the streamlines are parallel and for flow between two parallel surfaces we may consider the flow as made up of parallel laminar layers. In a pipe these laminar layers are cylindrical and may be called ***stream tubes***. In laminar flow, no mixing occurs between adjacent layers and it occurs at low average velocities.

3.2.2 TURBULENT FLOW

The shearing process causes energy loss and heating of the fluid. This increases with mean velocity. When a certain critical velocity is exceeded, the streamlines break up and mixing of the fluid occurs. The diagram illustrates Reynolds coloured ribbon experiment. Coloured dye is injected into a horizontal flow. When the flow is laminar the dye passes along without mixing with the water. When the speed of the flow is increased turbulence sets in and the dye mixes with the surrounding water. One explanation of this transition is that it is necessary to change the pressure loss into other forms of energy such as angular kinetic energy as indicated by small eddies in the flow.

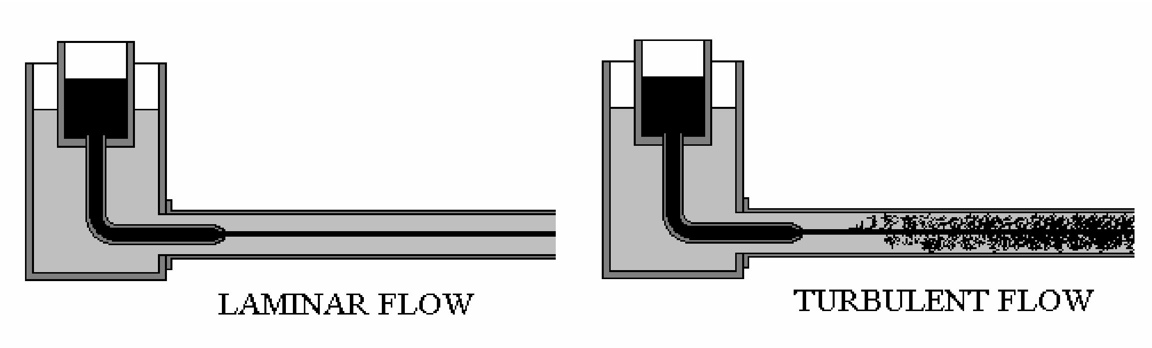


Fig.3.6

3.2.3 LAMINAR AND TURBULENT BOUNDARY LAYERS

In chapter 2 it was explained that a ***boundary layer*** is the layer in which the velocity grows from zero at the wall (no slip surface) to 99% of the maximum and the thickness of the layer is denoted δ. When the flow within the boundary layer becomes turbulent, the shape of the boundary layers waivers and when diagrams are drawn of turbulent boundary layers, the mean shape is usually shown. Comparing a laminar and turbulent boundary layer reveals that the turbulent layer is thinner than the laminar layer.

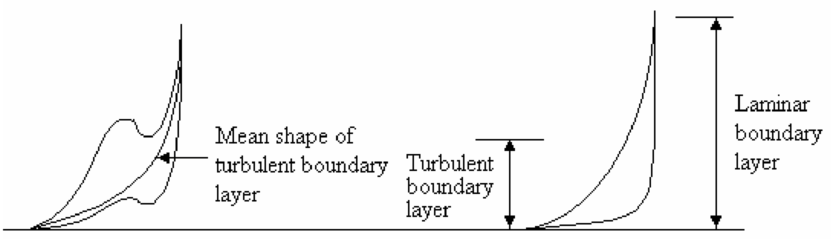


Fig.3.7

3.2.4 CRITICAL VELOCITY - REYNOLDS NUMBER

When a fluid flows in a pipe at a volumetric flow rate Q m^3/s the average velocity is defined $u_m = \frac{Q}{A}$ A is the cross sectional area.

The Reynolds number is defined as $R_e = \frac{\rho u_m D}{\mu} = \frac{u_m D}{\nu}$

If you check the units of R_e you will see that there are none and that it is a dimensionless number. You will learn more about such numbers in section ….?.

Reynolds discovered that it was possible to predict the velocity or flow rate at which the transition from laminar to turbulent flow occurred for any Newtonian fluid in any pipe. He also discovered that the critical velocity at which it changed back again was different. He found that when the flow was gradually increased, the change from laminar to turbulent always occurred at a Reynolds number of 2500 and when the flow was gradually reduced it changed back again at a Reynolds number of 2000. Normally, 2000 is taken as the critical value.

WORKED EXAMPLE 3.4

Oil of density 860 kg/m^3 has a kinematic viscosity of 40 cSt. Calculate the critical velocity when it flows in a pipe 50 mm bore diameter.

SOLUTION

$$R_e = \frac{u_m D}{\nu}$$

$$u_m = \frac{R_e \nu}{D} = \frac{2000 \text{x} 40 \text{x} 10^{-6}}{0.05} = 1.6 \text{ m/s}$$

3.3 DERIVATION OF POISEUILLE'S EQUATION for LAMINAR FLOW

Poiseuille did the original derivation shown below which relates pressure loss in a pipe to the velocity and viscosity for LAMINAR FLOW. His equation is the basis for measurement of viscosity hence his name has been used for the unit of viscosity. Consider a pipe with laminar flow in it. Consider a stream tube of length ΔL at radius r and thickness dr.

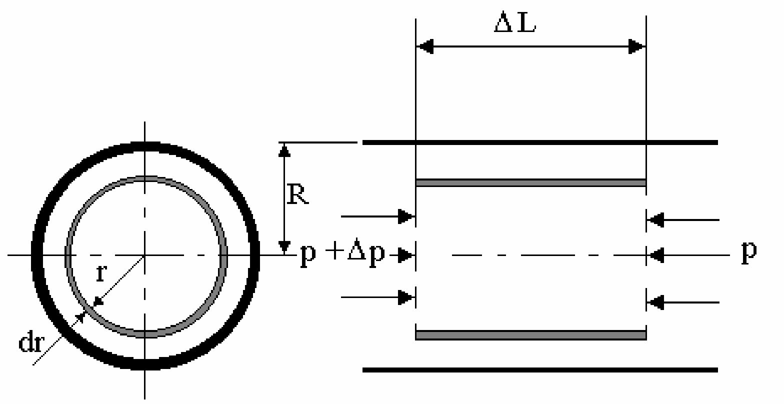


Fig.3.8

y is the distance from the pipe wall. $y = R - r$ $dy = -dr$ $\frac{du}{dy} = -\frac{du}{dr}$

The shear stress on the outside of the stream tube is τ. The force (F_s) acting from right to left is due to the shear stress and is found by multiplying τ by the surface area.

$Fs = \tau \times 2\pi r\,\Delta L$

For a Newtonian fluid , $\tau = \mu\frac{du}{dy} = -\mu\frac{du}{dr}$. Substituting for τ we get the following.

$$F_s = -2\pi r\Delta L\mu\frac{du}{dr}$$

The pressure difference between the left end and the right end of the section is Δp. The force due to this (F_p) is Δp x circular area of radius r.

$F_p = \Delta p \times \pi r^2$

Equating forces we have $-2\,\pi\,r\,\mu\,\Delta L\frac{du}{dr} = \Delta p\,\pi\,r^2$

$$du = -\frac{\Delta p}{2\,\mu\,\Delta L}rdr$$

In order to obtain the velocity of the streamline at any radius r we must integrate between the limits $u = 0$ when $r = R$ and $u = u$ when $r = r$.

$$\int_0^u du = -\frac{\Delta p}{2\,\mu\,\Delta L}\int_R^r rdr$$

$$u = -\frac{\Delta p}{4\,\mu\,\Delta L}\left(r^2 - R^2\right)$$

$$u = \frac{\Delta p}{4\,\mu\,L}\left(R^2 - r^2\right)$$

This is the equation of a Parabola so if the equation is plotted to show the boundary layer, it is seen to extend from zero at the edge to a maximum at the middle.

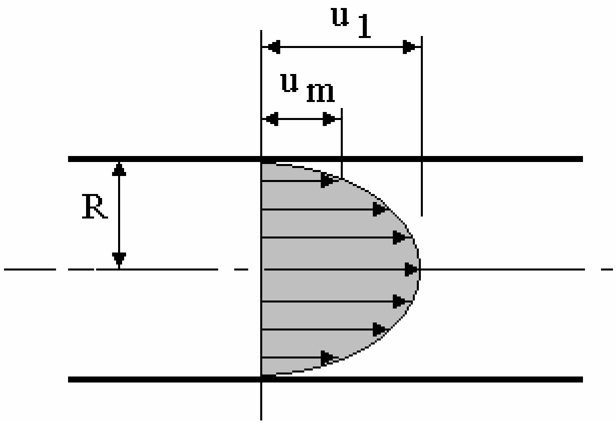


Fig.3.9

For maximum velocity put r = 0 and we get $u_1 = \dfrac{\Delta p\, R^2}{4\,\mu\,\Delta L}$

The average height of a parabola is half the maximum value so the average velocity is

$$u_m = \frac{\Delta p\, R^2}{8\,\mu\,\Delta L}$$

Often we wish to calculate the pressure drop in terms of diameter D. Substitute R=D/2 and rearrange.

$$\Delta p = \frac{32\,\mu\,\Delta L\, u_m}{D^2}$$

The volume flow rate is average velocity x cross sectional area.

$$Q = \frac{\pi R^2 \Delta p\, R^2}{8\,\mu\,\Delta L} = \frac{\pi R^4 \Delta p}{8\,\mu\,\Delta L} = \frac{\pi D^4 \Delta p}{128\,\mu\,\Delta L}$$

This is often changed to give the pressure drop as a friction head.

The friction head for a length L is found from $h_f = \Delta p/\rho g$ $$h_f = \frac{32\,\mu\, L\, u_m}{\rho\, g D^2}$$

This is Poiseuille's equation that applies only to laminar flow.

WORKED EXAMPLE 3.5

A capillary tube is 30 mm long and 1 mm bore. The head required to produce a flow rate of 8 mm^3/s is 30 mm. The fluid density is 800 kg/m^3.

Calculate the dynamic and kinematic viscosity of the oil.

SOLUTION

Rearranging Poiseuille's equation we get

$$\mu = \frac{h_f \rho g D^2}{32 L u_m}$$

$$A = \frac{\pi d^2}{4} = \frac{\pi \times 1^2}{4} = 0.785 \text{ mm}^2$$

$$u_m = \frac{Q}{A} = \frac{8}{0.785} = 10.18 \text{ mm/s}$$

$$\mu = \frac{0.03 \times 800 \times 9.81 \times 0.001^2}{32 \times 0.03 \times 0.01018} = 0.0241 \text{ N s/m or } 24.1 \text{ cP}$$

$$\nu = \frac{\mu}{\rho} = \frac{0.0241}{800} = 30.11 \times 10^{-6} \text{ m}^2/\text{s} \text{ or } 30.11 \text{ cSt}$$

WORKED EXAMPLE No.3.6

Oil flows in a pipe 100 mm bore with a Reynolds number of 250. The dynamic viscosity is 0.018 Ns/m^2. The density is 900 kg/m^3.

Determine the pressure drop per metre length, the average velocity and the radius at which it occurs.

SOLUTION

$Re=\rho u_m D/\mu$.

Hence

$u_m = Re\ \mu/\ \rho D$

$u_m = (250 \times 0.018)/(900 \times 0.1) = 0.05$ m/s

$\Delta p = 32\mu L\ u_m /D^2$

$\Delta p = 32 \times 0.018 \times 1 \times 0.05/0.1^2$

Δp= 2.88 Pascals.

$u = \{\Delta p/4L\mu\}(R^2 - r^2)$ which is made equal to the average velocity 0.05 m/s

$0.05 = (2.88/4 \times 1 \times 0.018)(0.05^2 - r^2)$

r = 0.035 m or 35.3 mm.

SELF ASSESSMENT EXERCISE 3.2

1. Oil flows in a pipe 80 mm bore diameter with a mean velocity of 0.4 m/s. The density is 890 kg/m^3 and the viscosity is 0.075 Ns/m^2.

   Show that the flow is laminar and hence deduce the pressure loss per metre length. (150 Pa)

2 Calculate the maximum velocity of water that can flow in laminar form in a pipe 20 m long and 60 mm bore. Determine the pressure loss in this condition. The density is 1000 kg/m^3 and the dynamic viscosity is 0.001 N s/m^2. (0.0333 m/s and 5.92 Pa)

3. Oil flow in a pipe 100 mm bore diameter with a Reynolds Number of 500. The density is 800 kg/m^3. The dynamic viscosity μ = 0.08 Ns/m^2.

   Calculate the velocity of a streamline at a radius of 40 mm. (0.36 m/s)

$$-\frac{dp}{dx}=\mu\frac{d^2u}{dy^2} \quad 4a$$

When a viscous fluid is subjected to an applied pressure it flows through a narrow horizontal passage as shown below. By considering the forces acting on the fluid element and assuming steady fully developed laminar flow, show that the velocity distribution is given by

b. Using the above equation show that for flow between two flat parallel horizontal surfaces distance t apart the velocity at any point is given by the following formula.

$$\mathbf{u = (1/2\mu)(dp/dx)(y^2 - ty)}$$

c. Carry on the derivation and show that the volume flow rate through a gap of height 't' and width 'B' is given by $Q=-B\frac{dp}{dx}\frac{t^3}{12\mu}$.

d. Show that the mean velocity 'u_m' through the gap is given by $u_m=-\frac{1}{12\mu}\frac{dp}{dx}t^2$

5 The volumetric flow rate of glycerine between two flat parallel horizontal surfaces 1 mm apart and 10 cm wide is 2 cm^3/s. Determine the following.

   i. the applied pressure gradient dp/dx. (240 kPa per metre)
   ii. the maximum velocity. (0.06 m/s)

   For glycerine assume that μ= 1.0 Ns/m^2 and the density is 1260 kg/m^3.

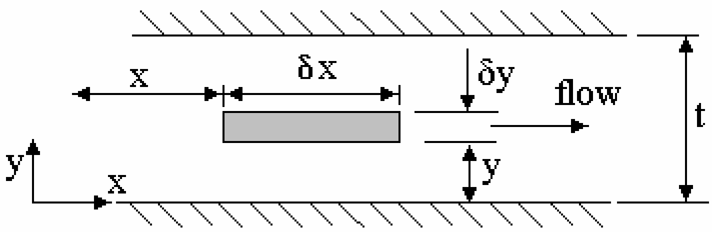


Fig.3.10

3.4 FRICTION COEFFICIENT

The friction coefficient is a convenient idea that can be used to calculate the pressure drop in a pipe. It is defined as follows.

$$C_f = \frac{\text{Wall Shear Stress}}{\text{Dynamic Pressure}}$$

3.4.1 DYNAMIC PRESSURE

Consider a fluid flowing with mean velocity u_m. If the kinetic energy of the fluid is converted into flow or fluid energy, the pressure would increase. The pressure rise due to this conversion is called the dynamic pressure.

$KE = ½\, mu_m^2$

Flow Energy = p Q   Q is the volume flow rate and $\rho = m/Q$

Equating $½\, mu_m^2 = p\,Q$ $p = mu^2/2Q = ½\, \rho\, u_m^2$

3.4.2 WALL SHEAR STRESS τ_o

The wall shear stress is the shear stress in the layer of fluid next to the wall of the pipe.

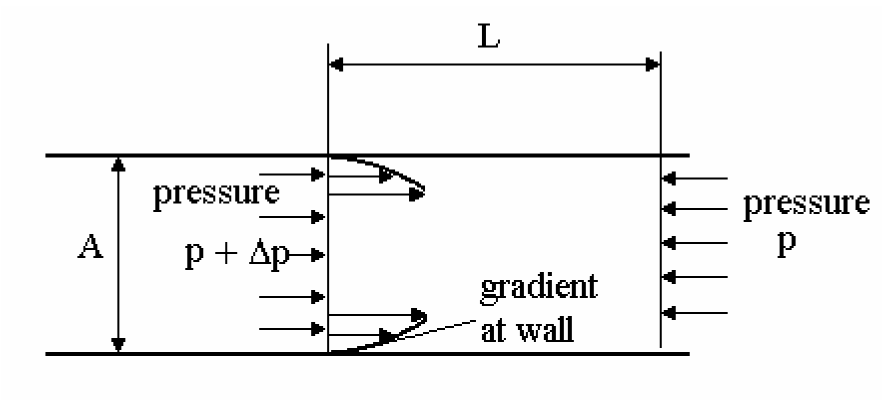


Fig.3.11

The shear stress in the layer next to the wall is $\tau_o = \mu\left(\frac{du}{dy}\right)_{wall}$

The shear force resisting flow is $F_s = \tau_o \pi LD$

The resulting pressure drop produces a force of $F_p = \frac{\Delta p \pi D^2}{4}$

Equating forces gives $\tau_o = \frac{D\,\Delta p}{4L}$

3.4.3 FRICTION COEFFICIENT for LAMINAR FLOW

$$C_f = \frac{\text{Wall Shear Stress}}{\text{Dynamic Pressure}} = \frac{2D\Delta p}{4L\rho u_m^2}$$

From Poiseuille's equation $\Delta p = \frac{32\mu L u_m}{D^2}$ Hence $C_f = \left(\frac{2D}{4L\rho u_m^2}\right)\left(\frac{32\mu L u}{D^2}\right) = \frac{16\mu}{\rho u_m^2 D} = \frac{16}{R_e}$

3.5 DARCY FORMULA

This formula is mainly used for calculating the pressure loss in a pipe due to turbulent flow but it can be used for laminar flow also.

Turbulent flow in pipes occurs when the Reynolds Number exceeds 2500 but this is not a clear point so 3000 is used to be sure. In order to calculate the frictional losses we use the concept of friction coefficient symbol C_f. This was defined as follows.

$$C_f = \frac{\text{Wall Shear Stress}}{\text{Dynamic Pressure}} = \frac{2D\Delta p}{4L\rho u_m^2}$$

Rearranging equation to make Δp the subject

$$\Delta p = \frac{4C_f L\rho u_m^2}{2D}$$

This is often expressed as a friction head h_f

$$h_f = \frac{\Delta p}{\rho g} = \frac{4C_f L u_m^2}{2gD}$$

This is the Darcy formula. In the case of laminar flow, Darcy's and Poiseuille's equations must give the same result so equating them gives

$$\frac{4C_f L u_m^2}{2gD} = \frac{32\mu L u_m}{\rho g D^2}$$

$$C_f = \frac{16\mu}{\rho u_m D} = \frac{16}{R_e}$$

This is the same result as before for laminar flow.

3.5.1 FLUID RESISTANCE

The above equations may be expressed in terms of flow rate Q by substituting u = Q/A

$$h_f = \frac{4C_f L u_m^2}{2gD} = \frac{4C_f LQ^2}{2gDA^2}$$

Substituting $A = \pi D^2/4$ we get the following.

$$h_f = \frac{32C_f LQ^2}{g\pi^2 D^5} = RQ^2$$

R is the fluid resistance or restriction. $R = \dfrac{32\,C_f\,L}{g\,\pi^2\,D^5}$

If we want pressure loss instead of head loss the equations are as follows.

$$p_f = \rho g h_f = \frac{32\rho C_f LQ^2}{\pi^2 D^5} = RQ^2$$

R is the fluid resistance or restriction. $R = \dfrac{32\,\rho\,C_f L}{\pi^2\,D^5}$

It should be noted that R contains the friction coefficient and this is a variable with velocity and surface roughness so R should be used with care.

3.5.2 MOODY DIAGRAM AND RELATIVE SURFACE ROUGHNESS

In general the friction head is some function of u_m such that $h_f = \phi u_m{}^n$. Clearly for laminar flow, n =1 but for turbulent flow n is between 1 and 2 and its precise value depends upon the roughness of the pipe surface. Surface roughness promotes turbulence and the effect is shown in the following work.

Relative surface roughness is defined as $\varepsilon = k/D$ where k is the mean surface roughness and D the bore diameter.

An American Engineer called Moody conducted exhaustive experiments and came up with the Moody Chart. The chart is a plot of C_f vertically against R_e horizontally for various values of ε. In order to use this chart you must know two of the three co-ordinates in order to pick out the point on the chart and hence pick out the unknown third co-ordinate. For smooth pipes, (the bottom curve on the diagram), various formulae have been derived such as those by Blasius and Lee.

BLASIUS $C_f = 0.0791\ R_e{}^{0.25}$

LEE $C_f = 0.0018 + 0.152\ R_e{}^{0.35}$.

The Moody diagram shows that the friction coefficient reduces with Reynolds number but at a certain point, it becomes constant. When this point is reached, the flow is said to be fully developed turbulent flow. This point occurs at lower Reynolds numbers for rough pipes.

A formula that gives an approximate answer for any surface roughness is that given by Haaland.

$$\frac{1}{\sqrt{C_f}} = -3.6\log_{10}\left\{\frac{6.9}{R_e} + \left(\frac{\varepsilon}{3.71}\right)^{1.11}\right\}$$

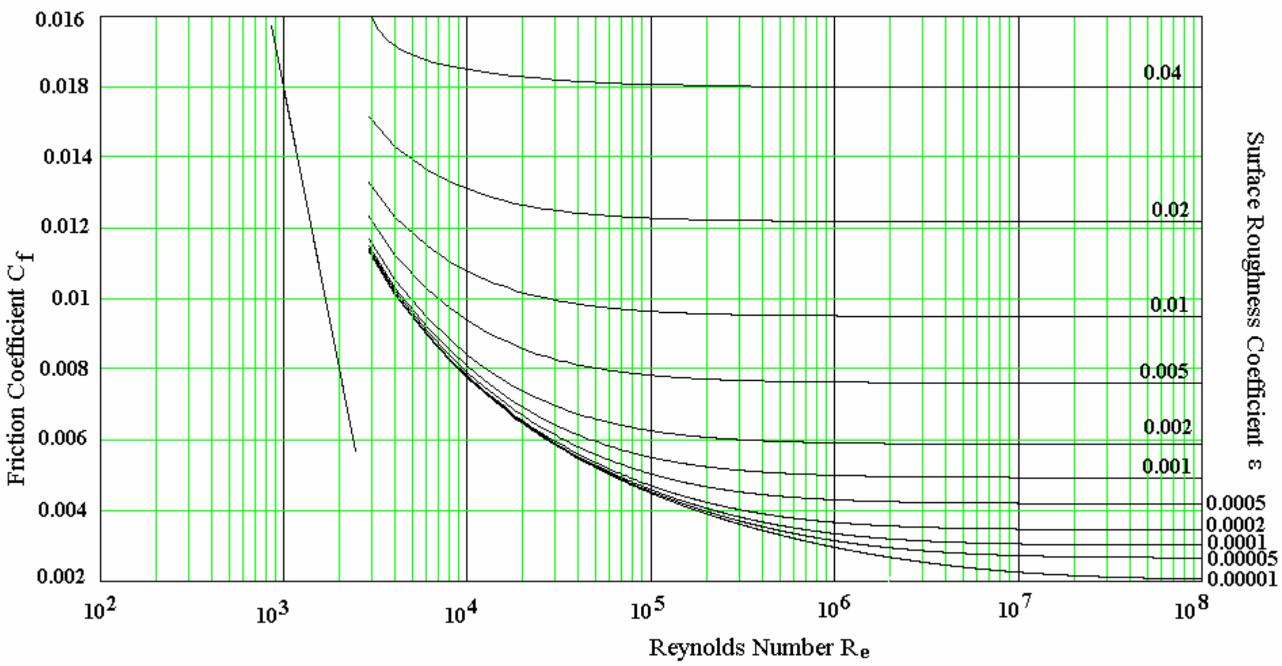


Fig.3.12 CHART

WORKED EXAMPLE 3.7

Determine the friction coefficient for a pipe 100 mm bore with a mean surface roughness of 0.06 mm when a fluid flows through it with a Reynolds number of 20 000.

SOLUTION

The mean surface roughness $\varepsilon = k/d = 0.06/100 = 0.0006$
Locate the line for $\varepsilon = k/d = 0.0006$.
Trace the line until it meets the vertical line at Re = 20 000. Read of the value of C_f horizontally on the left. Answer $C_f = 0.0067$

Check using the formula from Haaland.

$$\frac{1}{\sqrt{C_f}} = -3.6\log_{10}\left\{\frac{6.9}{R_e} + \left(\frac{\varepsilon}{3.71}\right)^{1.11}\right\}$$

$$\frac{1}{\sqrt{C_f}} = -3.6\log_{10}\left\{\frac{6.9}{20000} + \left(\frac{0.0006}{3.71}\right)^{1.11}\right\}$$

$$\frac{1}{\sqrt{C_f}} = -3.6\log_{10}\left\{\frac{6.9}{20000} + \left(\frac{0.0006}{3.71}\right)^{1.11}\right\}$$

$$\frac{1}{\sqrt{C_f}} = 12.206$$

$$C_f = 0.0067$$

WORKED EXAMPLE 3.8

Oil flows in a pipe 80 mm bore with a mean velocity of 4 m/s. The mean surface roughness is 0.02 mm and the length is 60 m. The dynamic viscosity is 0.005 N s/m^2 and the density is 900 kg/m^3. Determine the pressure loss.

SOLUTION

$Re = \rho ud/\mu = (900 \times 4 \times 0.08)/0.005 = 57600$

$\varepsilon = k/d = 0.02/80 = 0.00025$

From the chart $C_f = 0.0052$

$h_f = 4C_fLu2/2dg = (4 \times 0.0052 \times 60 \times 4^2)/(2 \times 9.81 \times 0.08) = 12.72$ m

$\Delta p = \rho g h_f = 900 \times 9.81 \times 12.72 = 112.32$ kPa.

SELF ASSESSMENT EXERCISE 3.3

1. A pipe is 25 km long and 80 mm bore diameter. The mean surface roughness is 0.03 mm. It caries oil of density 825 kg/m^3 at a rate of 10 kg/s. The dynamic viscosity is 0.025 N s/m^2.

   Determine the friction coefficient using the Moody Chart and calculate the friction head. (Ans. 3075 m.)

2. Water flows in a pipe at 0.015 m^3/s. The pipe is 50 mm bore diameter. The pressure drop is 13 420 Pa per metre length. The density is 1000 kg/m^3 and the dynamic viscosity is 0.001 N s/m^2.

   Determine the following.
   - i. The wall shear stress (167.75 Pa)
   - ii. The dynamic pressures (29180 Pa).
   - iii. The friction coefficient (0.00575)

   iv. The mean surface roughness (0.0875 mm)

3. Explain briefly what is meant by fully developed laminar flow. The velocity u at any radius r in fully developed laminar flow through a straight horizontal pipe of internal radius r_o is given by

   $$\mathbf{u = (1/4\mu)(r_o^2 - r^2)dp/dx}$$

   dp/dx is the pressure gradient in the direction of flow and μ is the dynamic viscosity. The wall skin friction coefficient is defined as $C_f = 2\tau_o/(\rho u_m^2)$.

   Show that $C_f = 16/R_e$ where $R_e = \rho u_m D/\mu$ an ρ is the density, u_m is the mean velocity and τ_o is the wall shear stress.

4. Oil with viscosity 2 x 10^{-2} Ns/m^2 and density 850 kg/m^3 is pumped along a straight horizontal pipe with a flow rate of 5 dm^3/s. The static pressure difference between two tapping points 10 m apart is 80 N/m^2. Assuming laminar flow determine the following.

   - i. The pipe diameter.
   - ii. The Reynolds number.

   Comment on the validity of the assumption that the flow is laminar.

3.6 MINOR LOSSES

Minor losses occur in the following circumstances.

i. Exit from a pipe into a tank.
ii. Entry to a pipe from a tank.
iii. Sudden enlargement in a pipe.
iv. Sudden contraction in a pipe.
v. Bends in a pipe.
vi. Any other source of restriction such as pipe fittings and valves.

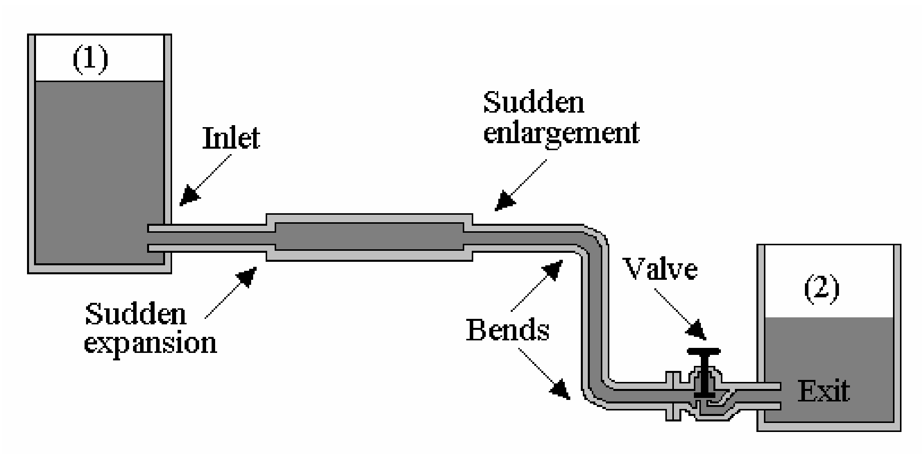


Fig.3.13

In general, minor losses are neglected when the pipe friction is large in comparison but for short pipe systems with bends, fittings and changes in section, the minor losses is the dominant factor.

In general, the minor losses are expressed as a fraction of the kinetic head or dynamic pressure in the smaller pipe.

Minor head loss = k $u^2/2g$ Minor pressure loss = ½ $k\rho u^2$

Values of k can be derived for standard cases but for items like elbows and valves in a pipeline, it is determined by experimental methods.

Minor losses can also be expressed in terms of fluid resistance R as follows.

$$h_L = k\frac{u^2}{2} = k\frac{Q^2}{2A^2} = k\frac{8Q^2}{\pi^2 D^4} = RQ^2 \quad \text{Hence } R = \frac{8k}{\pi^2 D^4}$$

$$p_L = k\frac{8\rho g Q^2}{\pi^2 D^4} = RQ^2 \text{ hence } R = \frac{8k\rho g}{\pi^2 D^4}$$

Before you go on to look at the derivations, you must first learn about the coefficients of contraction and velocity.

3.6.1 COEFFICIENT OF CONTRACTION Cc

The fluid approaches the entrance from all directions and the radial velocity causes the jet to contract just inside the pipe. The jet then spreads out to fill the pipe. The point where the jet is smallest is called the *VENA CONTRACTA*.

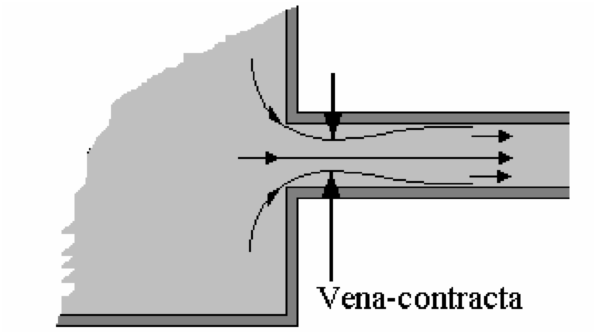


Fig.3.14

The coefficient of contraction C_c is defined as $C_c = A_j/A_o$

A_j is the cross sectional area of the jet and A_o is the c.s.a. of the pipe. For a round pipe this becomes $C_c = d_j^2/d_o^2$.

3.6.2 COEFFICIENT OF VELOCITY C_v

The coefficient of velocity is defined as **C_v = actual velocity/theoretical velocity**

In this instance it refers to the velocity at the vena-contracta but as you will see later on, it applies to other situations also.

3.6.3 EXIT FROM A PIPE INTO A TANK.

The liquid emerges from the pipe and collides with stationary liquid causing it to swirl about before finally coming to rest. All the kinetic energy is dissipated by friction. It follows that all the kinetic head is lost so k = 1.0

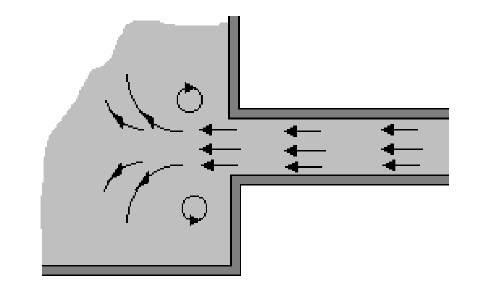


Fig.3.15

3.6.4 ENTRY TO A PIPE FROM A TANK

The value of k varies from 0.78 to 0.04 depending on the shape of the inlet. A good rounded inlet has a low value but the case shown is the worst.

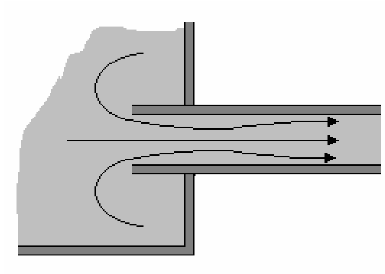


Fig.3.16

3.6.5 SUDDEN ENLARGEMENT

This is similar to a pipe discharging into a tank but this time it does not collide with static fluid but with slower moving fluid in the large pipe. The resulting loss coefficient is given by the following expression.

$$k = \left\{1 - \left(\frac{d_1}{d_2}\right)^2\right\}^2$$

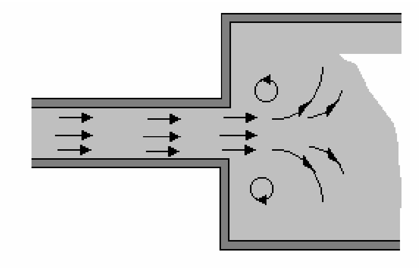


Fig.3.17

3.6.6 SUDDEN CONTRACTION

This is similar to the entry to a pipe from a tank. The best case gives k = 0 and the worse case is for a sharp corner which gives k = 0.5.

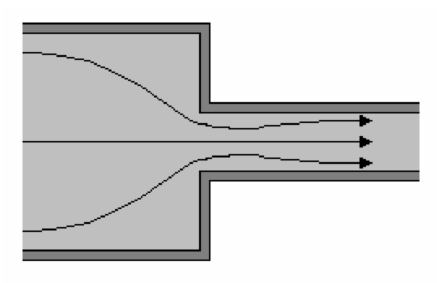


Fig.3.18

3.6.7 BENDS AND FITTINGS

The k value for bends depends upon the radius of the bend and the diameter of the pipe. The k value for bends and the other cases is on various data sheets. For fittings, the manufacturer usually gives the k value. Often instead of a k value, the loss is expressed as an equivalent length of straight pipe that is to be added to L in the Darcy formula.

WORKED EXAMPLE 3.9

A tank of water empties by gravity through a horizontal pipe into another tank. There is a sudden enlargement in the pipe as shown. At a certain time, the difference in levels is 3 m. Each pipe is 2 m long and has a friction coefficient $C_f = 0.005$. The inlet loss constant is K = 0.3.

Calculate the volume flow rate at this point.

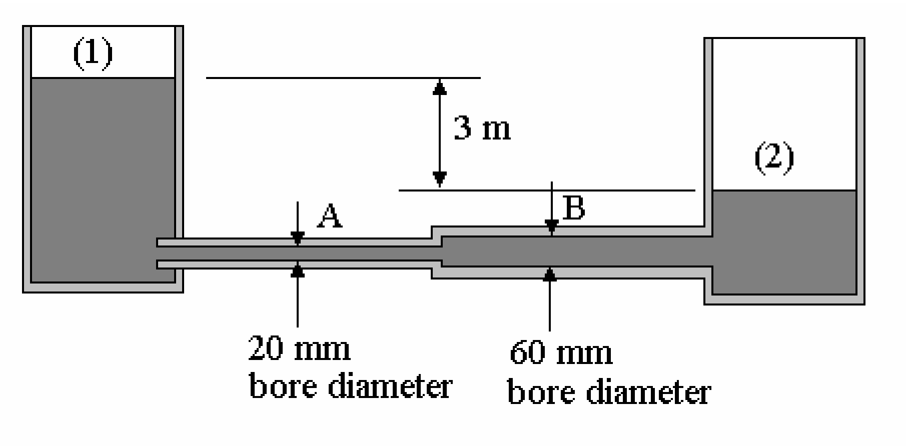


Fig.3.19

SOLUTION

There are five different sources of pressure loss in the system and these may be expressed in terms of the fluid resistance as follows.

The head loss is made up of five different parts. It is usual to express each as a fraction of the kinetic head as follows.

Resistance pipe A $\quad R_1 = \frac{32C_fL}{gD_A{}^5\pi^2} = \frac{32 \text{ x } 0.005 \text{ x } 2}{g \text{ x } 0.02^5\pi^2} = 1.0328 \text{ x } 10^6 \, s^2m^{-5}$

Resistance in pipe B $\quad R_2 = \frac{32C_fL}{gD_B^5\pi^2} = \frac{32 \text{ x } 0.005 \text{ x } 2}{g \text{ x } 0.06^5\pi^2} = 4.250\text{x}10^3 \, s^2m^{-5}$

Loss at entry K=0.3 $\quad R_3 = \frac{8K}{g\pi^2D_A{}^4} = \frac{8 \text{ x } 0.3}{g\,\pi^2 \text{x } 0.02^4} = 158 \, s^2m^{-5}$

Loss at sudden enlargement. $\quad k = \left\{1 - \left(\frac{d_A}{d_B}\right)^2\right\}^2 = \left\{1 - \left(\frac{20}{60}\right)^2\right\}^2 = 0.79$

$$R_4 = \frac{8K}{g\pi^2D_A{}^4} = \frac{8\text{x}0.79}{g\pi^2 \text{x } 0.02^4} = 407.7 \, s^2m^{-5}$$

Loss at exit K=1 $\quad R_5 = \frac{8K}{g\pi^2D_B{}^4} = \frac{8\text{x}1}{g\pi^2 \text{x } 0.06^4} = 63710 \, s^2m^{-5}$

Total losses.

$$h_L = R_1Q^2 + R_2Q^2 + R_3Q^2 + R_4Q^2 + R_5Q^2$$
$$h_L = (R_1 + R_2 + R_2 + R_4 + R_5)Q^2 = 1.101 \text{ x } 10^6 Q^2$$

BERNOULLI'S EQUATION

Apply Bernoulli between the free surfaces (1) and (2)

$$h_1 + z_1 + \frac{u_1^2}{2g} = h_2 + z_2 + \frac{u_2^2}{2g} + h_L$$

On the free surface the velocities are small and about equal and the pressures are both atmospheric so the equation reduces to the following.

$z_1 - z_2 = h_L = 3 \qquad 3 = 1.101 \text{ x } 10^6 \, Q^2$

$Q^2 = 2.724 \text{ x } 10^{-6} \qquad Q = 1.65 \text{ x } 10^{-3} \text{ m}^3/\text{s}$

3.7 SIPHONS

Liquid will siphon from a tank to a lower level even if the pipe connecting them rises above the level of both tanks as shown in the diagram. Calculation will reveal that the pressure at point (2) is lower than atmospheric pressure (a vacuum) and there is a limit to this pressure when the liquid starts to turn into vapour. For water about 8 metres is the practical limit that it can be sucked (8 m water head of vacuum).

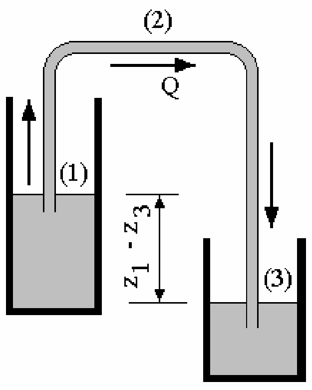


Fig.3.20

WORKED EXAMPLE 3.10

A tank of water empties by gravity through a siphon. The difference in levels is 3 m and the highest point of the siphon is 2 m above the top surface level and the length of pipe from inlet to the highest point is 2.5 m. The pipe has a bore of 25 mm and length 6 m. The friction coefficient for the pipe is 0.007.The inlet loss coefficient K is 0.7.

Calculate the volume flow rate and the pressure at the highest point in the pipe.

SOLUTION

There are three different sources of pressure loss in the system and these may be expressed in terms of the fluid resistance as follows.

Pipe Resistance $$R_1 = \frac{32C_f L}{gD^5\pi^2} = \frac{32 \text{ x } 0.007 \text{ x } 6}{g \text{ x } 0.025^5 \pi^2} = 1.422 \text{ x } 10^6 \text{ s}^2\text{m}^{-5}$$

Entry Loss Resistance $$R_2 = \frac{8K}{g\pi^2 D^4} = \frac{8 \text{ x } 0.7}{g\pi^2 \text{x } 0.025^4} = 15.1 \text{ x } 10^3 \text{ s}^2\text{m}^{-5}$$

Exit loss Resistance $$R_3 = \frac{8K}{g\pi^2 D^4} = \frac{8\text{x}1}{g\pi^2 \text{x } 0.025^4} = 21.57 \text{ x } 10^3 \text{ s}^2\text{m}^{-5}$$

Total Resistance $R_T = R_1 + R_2 + R_3 = 1.458 \text{ x } 10^6 \text{ s}^2 \text{ m}^{-5}$

Apply Bernoulli between the free surfaces (1) and (3)

$$h_1 + z_1 + \frac{u_1^2}{2g} = h_3 + z_3 + \frac{u_3^2}{2g} + h_L \qquad 0 + z_1 + 0 = 0 + z_3 + 0 + h_L \qquad z_1 - z_3 = h_L = 3$$

Flow rate $$Q = \sqrt{\frac{z_1 - z_3}{R_T}} = \sqrt{\frac{3}{1.458 \text{ x } 10^6}} = 1.434\text{x}10^{-3} \text{m}^3/\text{s}$$

Bore Area $A=\pi D^2/4 = \pi \text{ x } 0.025^2/4 = 490.87 \text{ x } 10^{-6} \text{ m}^2$

Velocity in Pipe $u = Q/A = 1.434 \text{ x } 10^{-3}/490.87 \text{ x } 10^{-6} = 2.922$ m/s

Apply Bernoulli between the free surfaces (1) and (2)

$$h_1 + z_1 + \frac{u_1^2}{2g} = h_2 + z_2 + \frac{u_2^2}{2g} + h_L \qquad 0 + 0 + 0 = h_2 + 2 + \frac{2.922^2}{2g} + h_L$$

$$h_2 = -2 - h_L \frac{2.922^2}{2g} - h_L \qquad h_2 = -2 + 0.435 - h_L = -2.435 - h_L$$

Calculate the losses between (1) and (2)

Pipe friction Resistance is proportionally smaller by the length ratio.

$R_1 = (2.5/6) \text{ x}1.422 \text{ x } 10^6 = 0.593 \text{ x } 10^6$

Entry Resistance $R_2 = 15.1 \text{ x } 10^3$ as before

Total resistance $R_T = 608.1 \text{ x } 10^3$

Head loss $h_L = R_T Q^2 = 1.245$m

The pressure head at point (2) is hence $h_2 = -2.435 - 1.245 = -3.68$ m head

3.8 MOMENTUM and PRESSURE FORCES

Changes in velocities mean changes in momentum and Newton's second law tells us that this is accompanied by a force such that

Force = rate of change of momentum.

Pressure changes in the fluid must also be considered as these also produce a force. Translated into a form that helps us solve the force produced on devices such as those considered here, we use the equation $F = \Delta(pA) + m\,\Delta u$.

When dealing with devices that produce a change in direction, such as pipe bends, this has to be considered more carefully and this is covered in chapter 4. In the case of sudden changes in section, we may apply the formula

$F = (p_1A_1 + mu_1) - (p_2 A_2 + mu_2)$ point 1 is upstream and point 2 is downstream.

WORKED EXAMPLE 3.11

A pipe carrying water experiences a sudden reduction in area as shown. The area at point (1) is 0.002 m^2 and at point (2) it is 0.001 m^2. The pressure at point (2) is 500 kPa and the velocity is 8 m/s. The loss coefficient K is 0.4. The density of water is 1000 kg/m^3. Calculate the following.

i. The mass flow rate.
ii. The pressure at point (1)
iii. The force acting on the section.

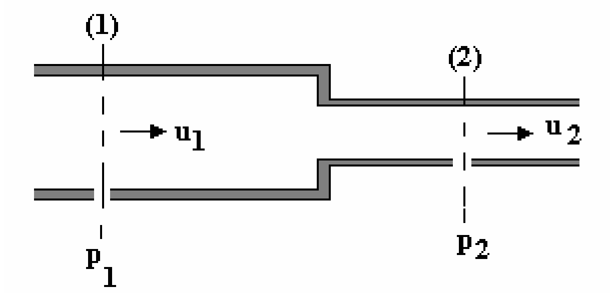


Fig.3.21

SOLUTION

$u_1 = u_2A_2/A_1 = (8 \text{ x } 0.001)/0.002 = 4$ m/s
$m = \rho A_1u_1 = 1000 \text{ x } 0.002 \text{ x } 4 = 8$ kg/s.
$Q = A_1u_1 = 0.002 \text{ x } 4 = 0.008 \text{ m}^3/\text{s}$
Pressure loss at contraction = $\frac{1}{2}\rho k u_1^2 = \frac{1}{2} \text{ x } 1000 \text{ x } 0.4 \text{ x } 4^2 = 3200$ Pa
Apply Bernoulli between (1) and (2)

$$p_1 + \frac{\rho u_1^2}{2} = p_2 + \frac{\rho u_2^2}{2} + p_L$$

$$p_1 + \frac{1000 \text{ x } 4^2}{2} = 500 \text{ x } 10^3 + \frac{1000 \text{ x } 8^2}{2} + 3200$$

$$p_1 = 527.2 \text{ kPa}$$

$F = (p_1A_1 + mu_1) - (p_2 A_2 + mu_2)$

$F = [(527.2 \text{ x } 10^3 \text{ x } 0.002) + (8 \text{ x } 4)] – [500 \text{ x } 10^3 \text{ x } 0.001) + (8 \text{ x } 8)]$

$F = 1054.4 + 32 – 500 – 64$

$F = 522.4$ N

SELF ASSESSMENT EXERCISE 3.4

1. A pipe carries oil at a mean velocity of 6 m/s. The pipe is 5 km long and 1.5 m diameter. The surface roughness is 0.8 mm. The density is 890 kg/m^3 and the dynamic viscosity is 0.014 N s/m^2. Determine the friction coefficient from the Moody chart and go on to calculate the friction head h_f. (Ans. $C_f = 0.0045$ $h_f = 110.1$ m)

2. The diagram shows a tank draining into another lower tank through a pipe. Note the velocity and pressure is both zero on the surface on a large tank. Calculate the flow rate using the data given on the diagram. (Ans. 7.16 dm^3/s)

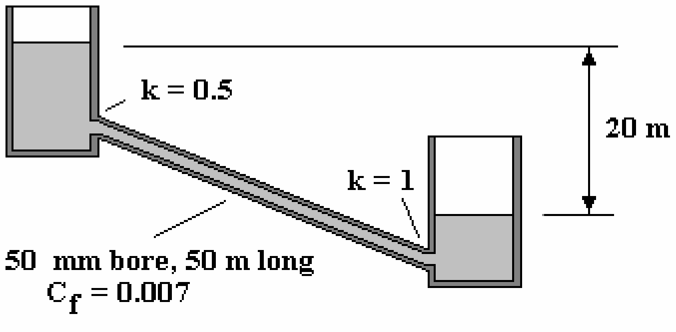


Fig.3.22

3. Water flows through the sudden pipe expansion shown below at a flow rate of 3 dm^3/s. Upstream of the expansion the pipe diameter is 25 mm and downstream the diameter is 40 mm. There are pressure tappings at section (1), about half a diameter upstream, and at section (2), about 5 diameters downstream. At section (1) the gauge pressure is 0.3 bar.
   Evaluate the following.
   (i) The gauge pressure at section (2) (0.387 bar)
   (ii) The total force exerted by the fluid on the expansion. (-23 N)

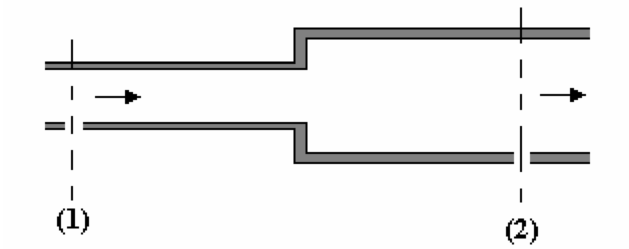


Fig.3.23

4. A domestic water supply consists of a large tank with a loss free-inlet to a 10 mm diameter pipe of length 20 m, that contains 9 right angles bends. The pipe discharges to atmosphere 8.0 m below the free surface level of the water in the tank.

   Evaluate the flow rate of water assuming that there is a loss of 0.75 velocity heads in each bend and that friction in the pipe is given by the Blasius equation $C_f=0.079(Re)^{-0.25}$ (0.118 dm^3/s). The dynamic viscosity is 0.89 x 10^{-3} and the density is 997 kg/m^3.

5. A tank of water empties by gravity through a siphon into a lower tank. The difference in levels is 6 m and the highest point of the siphon is 2 m above the top surface level. The length of pipe from the inlet to the highest point is 3 m. The pipe has a bore of 30 mm and length 11 m. The friction coefficient for the pipe is 0.006.The inlet loss coefficient K is 0.6.

   Calculate the volume flow rate and the pressure at the highest point in the pipe.
   (Answers 2.378 dm^3/s and –4.31 m)

3.10.1 WATER HAMMER

In this section, we will examine the causes of water hammer. The sudden acceleration or deceleration of fluids in pipes is accompanied by corresponding changes in pressure that can be extremely large. In the extreme, the pressure surge can split the pipe. The phenomenon is often accompanied by load hammer noises, hence the name.

First, we must examine the Bulk Modulus (K) and the derivation of the acoustic velocity in an elastic fluid.

3.10.1 BULK MODULUS (K)

Bulk modulus was discussed in Chapter 1 and defined as follows.

$$K = \frac{\text{Change in pressure}}{\text{Volumetric strain}} = \frac{V\Delta p}{\Delta V} = \frac{V\delta p}{\delta V}$$

V is volume and p is pressure. The following work shows how this may be changed to the form **K = ρdp/dρ**

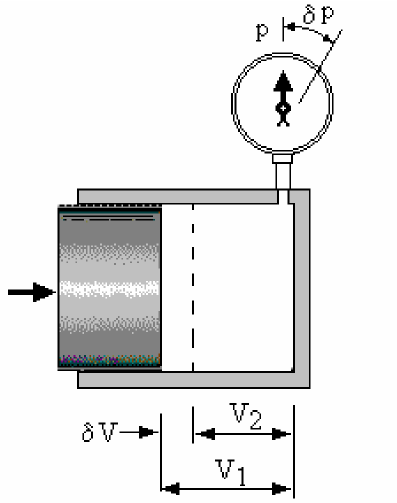


Fig.3.24

Consider a volume V_1 that is compressed to volume V_2 by a small increase in pressure δp. The reduction in volume is δV. The initial density is ρ and this increases by δρ

The mass of δV is $\delta m = \rho\,\delta V$ ....(3.10.1)
The initial mass of V_2 is $m_1 = \rho\, V_2$ ....(3.10.2)
The final mass of V_2 is $m_2 = (\rho + \delta\rho)\, V_2$.............(3.10.3)

The increase in mass is due to the mass of δV being compressed into the volume V_2. Hence (3.10.1) = (3.10.3) - (3.10.2)

$$\rho\,\delta V = (\rho + \delta\rho)\,V_2 - \rho V_2 = \rho\,V_2 + \delta\rho\,V_2 - \rho V_2$$
$$\rho\,\delta V = \delta\rho\,V_2 = \delta\rho\,(V_1 - \delta V) = V_1\delta\rho - \delta\rho\,\delta V$$

The product of two small quantities (δρ δV) is infinitesimally small so it may be ignored.

$$\rho\delta V = V_1\delta\rho \qquad \frac{\delta V}{V_1} = \frac{\delta\rho}{\rho}$$

$$\frac{V_1}{\delta V} = \frac{\rho}{\delta\rho}$$ substitute this into the formula for K

$$K = \frac{V\,\delta p}{\delta V} = \frac{\rho\delta p}{\delta\rho}$$

In the limit as δV → 0, we may revert to calculus notation.

Hence **K = ρdp/dρ**

3.10.2 SPEED OF SOUND IN AN ELASTIC MEDIUM

Most students don't need to know the derivation of the formula for the speed of sound but for those who are interested, here it is.

Consider a pipe of cross sectional area A full of fluid. Suppose a piston is pushed into the end with a velocity u m/s. Due to the compressibility of the fluid, further along the pipe at distance L, the fluid is still stationary. It has taken t seconds to achieve this position. The velocity of the interface is hence a = L/t m/s. In the same time the piston has moved x metres so u = x/t.

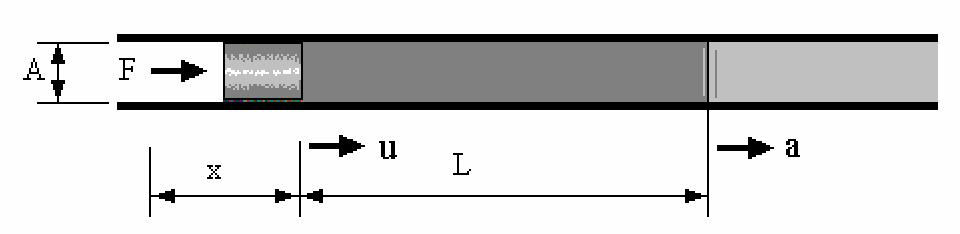


Fig. 3.25

The moving fluid has been accelerated from rest to velocity a. The inertia force needed to do this is in the form of pressure so the moving fluid is at a higher pressure than the static fluid and the interface is hence a pressure wave travelling along the pipe at velocity a.

The volume Ax has been compacted into the length L. The initial density of the fluid is ρ.
The mass compacted into length L is $dm = \rho Ax$.
substitute $x = ut$ $dm = \rho A u t$ ....(3.10.4)
The density of the compacted fluid has increased by $d\rho$ so the mass in the length L has increased by $dm = A L d\rho$

Substitute $L = at$ $dm = A a t d\rho$......(3.10.5)
Equate (3.10. 4) and (3.10.5) $\rho A u t= A a t d\rho$
$a = u \rho/d\rho$...........(3.10.6)
The force to accelerate the fluid from rest to a m/s is given by Newton's 2nd law
F = mass x acceleration = $A\, dp$
mass = ρAL acceleration = u/t
$A\, dp = \rho A L u/t$ $dp = \rho L u/t$
Substitute $L = at$ then $dp = \rho a u$ $a = (dp/u\rho)$ ....(3.10.7)

The velocity of the pressure wave a is by definition the acoustic velocity. Multiplying (3.10.8) by (3.11.7) gives a^2.

Hence $a^2=(u\rho/d\rho)(dp/u\rho)$ $a = (dp/d\rho)^{½}$ ....(3.10.9)

Previously it was shown that $K =\rho\, dp/d\rho$ $\mathbf{a =(K/\rho)^{½}}$

Students who have studied fundamental thermodynamics will understand the following extension of the theory to gases. The following section is not needed by those following the basic module.
Two important gas constants are the adiabatic index γ and the characteristic gas constant R. For a gas, the pressure change is adiabatic and if dp is small then the adiabatic law applies.
pV^{γ} = Constant
Dividing through by m^{γ} we get $p(V/m)^{\gamma} = \text{constant}/m^{\gamma} = \text{constant}$ $p/\rho^{\gamma} = C$
Differentiating we get $dp/d\rho = C\,(\gamma\rho^{\gamma-1})$ $dp/d\rho = (p/\rho^{\gamma})(\gamma\rho^{\gamma-1})$ $dp/d\rho = p\gamma/\rho$

from (3.8) it follows that $a = (p\gamma/\rho)^{½}$

from the gas law we have

$$pV = mRT$$
$$p = (m/V)RT$$
$$p = \rho RT$$

The velocity of a sound wave is that of a weak pressure wave. If the pressure change is large then $dp/d\rho$ is not a constant and the velocity would be that of a shock wave which is larger than the acoustic velocity.

For air $\gamma = 1.4$ and R = 287 J/kg K. Hence at 20°C (293 K) the acoustic velocity in air is as follows.

$$a = (\gamma RT)^{1/2} = (1.4 \times 287 \times 293)^{½} = 343 \text{ m/s}$$

3.10.3 PRESSURE SURGES DUE TO GRADUAL VALVE CLOSURE

Consider a pipe line with a fluid flowing at a steady velocity of u m/s. A stop valve is gradually closed thus decelerating the fluid uniformly from u to zero in t seconds.

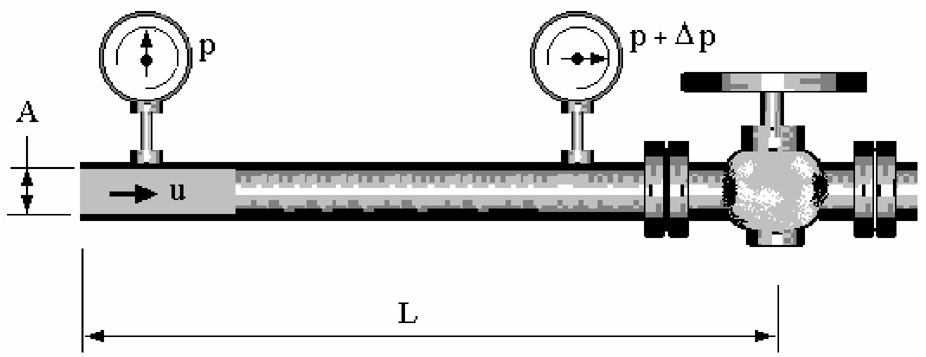


Fig.3.26

Volume of fluid = AL   Mass of fluid = ρAL Deceleration = u/t
Inertia force required F = mass x deceleration = $\rho AL\, u/t$

To provide this force the pressure of the fluid rises by Δp and the force is $A\,\Delta p$.

Equating forces we have $A\,\Delta p = \rho AL\, u/t$ $\mathbf{\Delta p = \rho L\, u/t}$

3.10.4 PRESSURE SURGES DUE TO SUDDEN VALVE CLOSURE

If the valve is closed suddenly then as t is very small the pressure rise is very large. In reality, a valve cannot close instantly but very rapid closure produces very large pressures. When this occurs, the compressibility of the fluid and the elasticity of the pipe is an important factor in reducing the rise in pressure. First, we will consider the pipe as rigid.

When the fluid stops suddenly at the valve, the fluid further up the pipe is still moving and compacting into the static fluid. An interface between moving and static fluid (a shock wave) travels up the pipe at the acoustic velocity. This is given by the equation :

$$a = (K/\rho)^{½} \qquad K = V\,dp/dV$$

If we assume that the change in volume is directly proportional to the change in pressure then we may change this to finite changes such that

$$K = V\,\delta p/\delta V \qquad\qquad \delta V = V\delta p/K$$

The mean pressure rise is $\delta p/2$
The strain energy stored by the compression = $\delta p\ \delta V/2$
The change in kinetic energy = ½ mu^2

Equating for energy conservation we get

$$Mu^2/ = \delta p\ \delta V/2 = (\delta p)V(\delta p)/2K$$
$$mu^2 = V(\delta p)^2/K$$
$$mKu^2/V = (\delta p)^2$$
$$(\delta p)^2 = (m/V)K\,u^2$$
$$(\delta p)^2 = \rho\,K\,u^2$$
$$\delta p = u(K\rho)^{½}$$

Since $a^2 = K/\rho$ then $K = a^2\rho$
$\delta p = u(a^2\rho^2)^{½}$
Then $\delta p = u\,a\,\rho$

For a large finite change, this becomes $\boldsymbol{\Delta p = a\,u\,\rho}$

<u>WORKED EXAMPLE 3.10</u>

A pipe 500 m long carries water at 2 m/s. Calculate the pressure rise produced when

a) the valve is closed uniformly in 5 seconds.
b) when it is shut suddenly.

The density of water is 1000 kg/m^3 and the bulk modulus is 4 GPa throughout.

<u>SOLUTION</u>

Uniform closure. $\Delta p = \rho\,L\,u/t = 1000 \times 500 \times 2/5 = 200$ kPa

Sudden closure $a = (K/\rho)^{½} = (4 \times 10^9/1000)^{½} = 2\,000$ m/s
$\Delta p = a\,u\,\rho = 2000 \times 2 \times 1000 = 4$ MPa

SELF ASSESSMENT EXERCISE 3.5

The density of water is 1000 kg/m^3 and the bulk modulus is 4 GPa throughout.

1. A pipe 50 m long carries water at 1.5 m/s. Calculate the pressure rise produced when

   a) the valve is closed uniformly in 3 seconds. (25 kPa)

   b) when it is shut suddenly. (3 MPa)

2. A pipe 2000 m long carries water at 0.8 m/s. A valve is closed. Calculate the pressure rise when

   a) it is closed uniformly in 10 seconds. (160 kPa)

   b) it is suddenly closed. (1.6 MPa)

3.10.5 THE EFFECT OF ELASTICITY IN THE PIPE

A pressure surge in an elastic pipe will cause the pipe to swell and some of the energy will be absorbed by straining the pipe wall. This reduces the rise in pressure. The more elastic the wall is, the less the pressure rise will be. Consider the case shown.

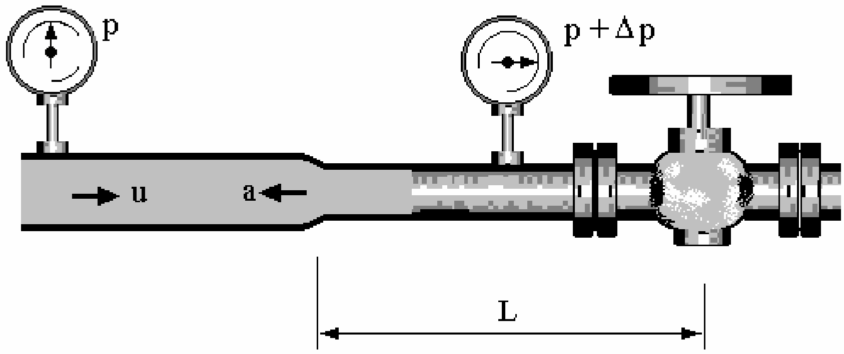
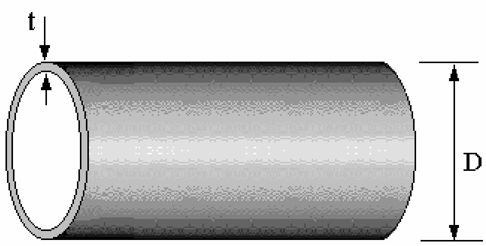


Fig.3.27

Kinetic Energy lost by fluid = ½ mu^2
The mass of fluid is ρAL so substituting K.E. = ½ ρALu^2

Strain Energy of fluid = $\Delta p^2 AL/2K$ (from last section)

Now consider the strain energy of the pipe wall. The strain energy of an elastic material with a direct stress σ is given by



S.E. = $(\sigma^2/2E)$ x volume of material

Fig.3.28

The pipe may be regarded as a thin cylinder and suitable references will show that stress stretching it around the circumference is given by the following formula.

$$\sigma = \Delta pD/2t$$

Volume of metal = πDtL Hence $\text{S.E.} = \left(\frac{\Delta p\, D}{2t}\right)^2 \text{x} \frac{\pi DtL}{2E} = \frac{(\Delta p)^2 DAL}{2tE}$

Equating KE lost to the total S.E. gained yields

$$\frac{\rho A L u^2}{2} = \frac{(\Delta p)^2 DAL}{2tE} + \frac{(\Delta p)^2 AL}{2K}$$

$$\frac{\rho u^2}{2} = \frac{(\Delta p)^2 D}{2tE} + \frac{(\Delta p)^2}{K} = \frac{(\Delta p)^2 D}{etE} + \frac{1}{K} \qquad \Delta p = \frac{\rho u}{\frac{D}{tE} + \frac{1}{K}}$$

The solution is usually given in terms of the effective bulk modulus K' which is defined as follows.

$$K' = \left\{\frac{D}{tE} + \frac{1}{K}\right\}^{-1}$$

The pressure rise is then given by **$\Delta p = u[\rho/K']^{1/2}$**

The acoustic velocity in an elastic pipe becomes a' and is given as $a' = (K'/\rho)^{1/2}$

Hence **$\Delta p = \rho\, u\, a'$**

WORKED EXAMPLE 3.11

A steel pipe carries water at 2 m/s. The pipe is 0.8 m bore diameter and has a wall 5 mm thick. Calculate the pressure rise produced when the flow is suddenly interrupted. The density of water is 1000 kg/m^3 and the bulk modulus is 4 GPa. The modulus of elasticity for steel E is 200 Gpa.

SOLUTION

$$K' = \left\{\frac{D}{tE} + \frac{1}{K}\right\}^{-1} = \left\{\frac{0.8}{0.005 \text{ x } 200 \text{ x } 10^9} + \frac{1}{4 \text{ x } 10^9}\right\}^{-1} = 952.4 \text{ MPa}$$

Sudden closure

$a' = (K'/\rho)^{1/2} = (952.4 \text{ x } 10^6/1000)^{1/2} = 976$ m/s

$\Delta p = a'\, u\, \rho = 976 \text{ x } 2 \text{ x } 1000 = 1.95$ MPa

3.10.6 DAMPING OUT PRESSURE SURGES

Pressure surges or water hammer occurs whenever there is a change in flow rate. There are many causes for this besides the opening and closing of valves. Changes in pump speeds may cause the same effect. Piston pumps in particular cause rapid acceleration and deceleration of the fluid. In power hydraulics, changes in the velocity of the ram cause the same effect. The problem occurs both on large scale plant such as hydroelectric pipelines and on small plant such as power hydraulic systems. The principles behind reduction of the pressure surges are the same for each, only the scale of the equipment is different.

For example, on power hydraulic systems, accumulators are used. These are vessels filled with both liquid and gas. On piston pumps, air vessels attached to the pipe are used. In both cases, a sudden rise in pressure produces compression of the gas that absorbs the strain energy and then releases it as the pressure passes.

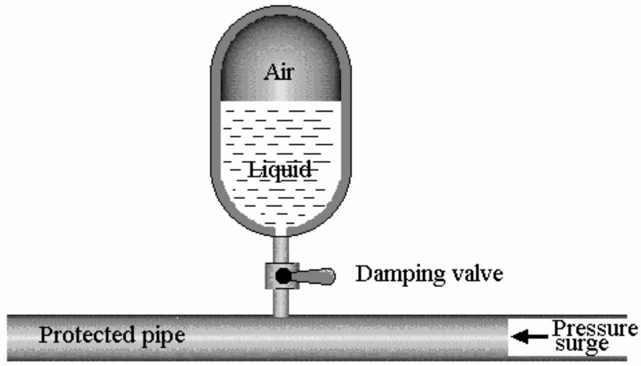


Fig.3.29

On hydroelectric schemes or large pumped systems, a surge tank is used. This is an elevated reservoir attached as close to the equipment needing protection as possible. When the valve is closed, the large quantity of water in the main system is diverted upwards into the surge tank. The pressure surge is converted into a raised level and hence potential energy. The level drops again as the surge passes and an oscillatory trend sets in with the water level rising and falling. A damping orifice in the pipe to the surge tank will help to dissipate the energy as friction and the oscillation dies away quickly.

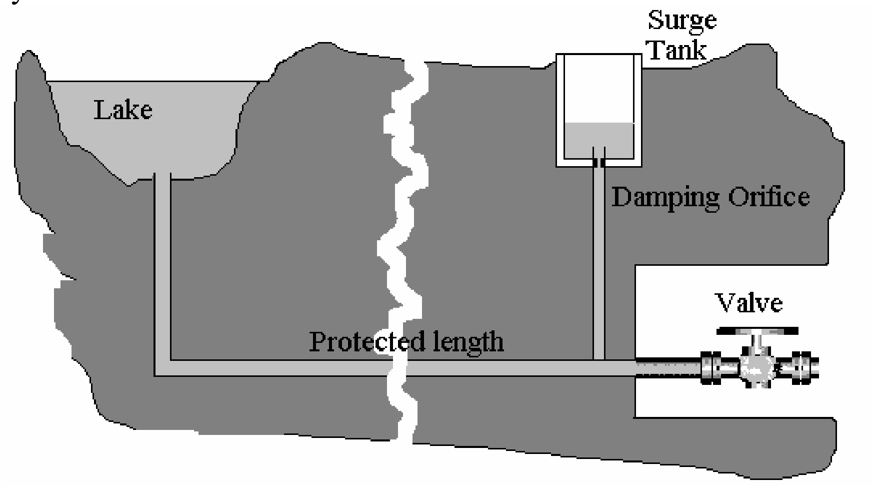


Fig.3.30

SELF ASSESSMENT EXERCISE 3.6

The density of water is 1000 kg/m^3 and the bulk modulus is 4 GPa throughout. The modulus of elasticity for steel E is 200 GPa.

1. A steel pipeline from a reservoir to a treatment works is 1 m bore diameter and has a wall 10 mm thick. It carries water with a mean velocity of 1.5 m/s. Calculate the pressure rise produced when the flow is suddenly interrupted. (1.732 MPa)

2. On a hydroelectric scheme, water from a high lake is brought down a vertical tunnel to a depth of 600 m and then connects to the turbine house by a horizontal high-pressure tunnel lined with concrete. The flow rate is 5 m^3/s and the tunnel is 4 m diameter.

   (i) Calculate the static pressure in the tunnel under normal operating conditions.(5.9 MPa)

   (ii) Explain the dangers to the high-pressure tunnel when the turbines are suddenly stopped.

   (iii) Assuming the tunnel wall is rigid, calculate the maximum pressure experienced in the high-pressure tunnel when flow is suddenly stopped. (6.7 MPa)

   (iv) Explain the safety features that are used in such situations to protect the tunnel.

EDEXCEL HIGHER

FLUID MECHANICS H1 UNIT 8

NQF LEVEL 4

OUTCOME 3 - THE FLOW OF REAL FLUIDS

TUTORIAL 4 - DRAG

3 **Flow of real fluids**

Head losses**:** head loss in pipes by Darcy's formula; Moody diagram; head loss due to sudden enlargement and contraction of pipe diameter; head loss at entrance to a pipe; head loss in valves; flow between reservoirs due to gravity; hydraulic gradient; siphons; hammer blow in pipes

Reynolds' number**:** inertia and viscous resistance forces; laminar and turbulent flow; critical velocities

Viscous drag**:** dynamic pressure; form drag; skin friction drag; drag coefficient

Dimensional analysis**:** checking validity of equations such as those for pressure at depth; thrust on immersed surfaces and impact of a jet; forecasting the form of possible equations such as those for Darcy's formula and critical velocity in pipes

This tutorial carries on from tutorial 3 and deals with how real fluids flow around bodies. When you have completed this tutorial you should be able to Explain how fluids exert a drag force on a body.

1. DRAG

When a fluid flows around the outside of a body, it produces a force that tends to drag the body in the direction of the flow. The drag acting on a moving object such as a ship or an aeroplane must be overcome by the propulsion system. Drag takes two forms, skin friction drag and form drag.

1.1 SKIN FRICTION DRAG

Skin friction drag is due to the viscous shearing that takes place between the surface and the layer of fluid immediately above it. This occurs on surfaces of objects that are long in the direction of flow compared to their height. Such bodies are called ***STREAMLINED***. When a fluid flows over a solid surface, the layer next to the surface may become attached to it (it wets the surface). This is called the 'no slip condition'. The layers of fluid above the surface are moving so there must be shearing taking place between the layers of the fluid. The shear stress acting between the wall and the first moving layer next to it is called ***the wall shear stress*** and denoted τ_w.

The result is that the velocity of the fluid grows from zero at the surface to a maximum u_o at some distance δ above it. This layer is called the ***BOUNDARY LAYER*** and δ is the boundary layer thickness. Fig. 1 Shows how the velocity "u" varies with height "y" for a typical boundary layer.

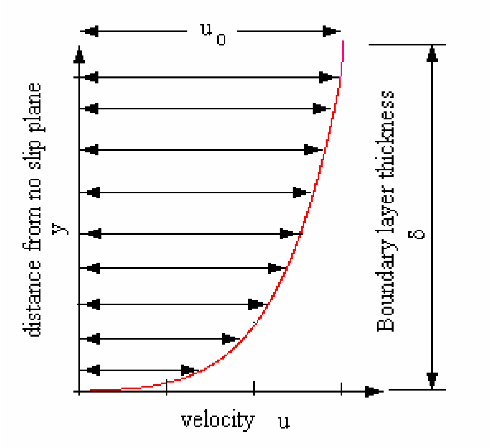


Fig. 1

In a pipe, this is the only form of drag and it results in a pressure and energy lost along the length. A thin flat plate is an example of a streamlined object. Consider a stream of fluid flowing with a uniform velocity u_o. When the stream is interrupted by the plate (fig. 2), the boundary layer forms on both sides. The diagram shows what happens on one side only.

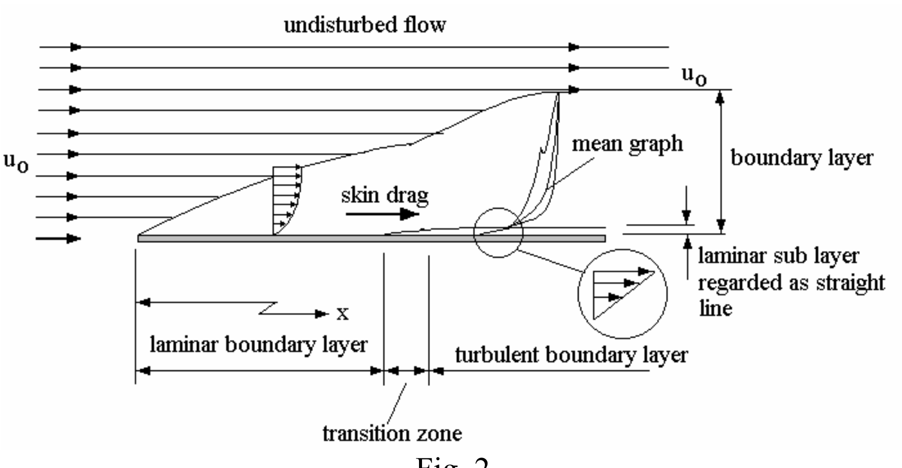


Fig. 2

The boundary layer thickness δ grows with distance from the leading edge. At some distance from the leading edge, it reaches a constant thickness. It is then called a ***FULLY DEVELOPED BOUNDARY LAYER***.

The Reynolds number for these cases is defined as: $$(R_e)_x = \frac{\rho u_o x}{\mu}$$

x is the distance from the leading edge. At low Reynolds numbers, the boundary layer may be laminar throughout the entire thickness. At higher Reynolds numbers, it is turbulent. This means that at some distance from the leading edge the flow within the boundary layer becomes turbulent.

A turbulent boundary layer is very unsteady and the streamlines do not remain parallel. The boundary layer shape represents an average of the velocity at any height. There is a region between the laminar and turbulent section where transition takes place

The turbulent boundary layer exists on top of a thin laminar layer called the ***LAMINAR SUB LAYER***. The velocity gradient within this layer is linear as shown. A deeper analysis would reveal that for long surfaces, the boundary layer is turbulent over most of the length. Many equations have been developed to describe the shape of the laminar and turbulent boundary layers and these may be used to estimate the skin friction drag.

Note that for this ideal example, it is assumed that the velocity is the undisturbed velocity u_o everywhere outside the boundary layer and that there is no acceleration and hence no change in the static pressure acting on the surface. There is hence no drag force due to pressure changes.

CALCULATING SKIN DRAG

The skin drag is due to the wall shear stress τ_w and this acts on the surface area (wetted area).
The drag force is hence: **R = τ_w x wetted area.** The dynamic pressure is the pressure resulting from the conversion of the kinetic energy of the stream into pressure and is defined by the expression $\frac{\rho u_o^2}{2}$.The drag coefficient is defined as

$$C_{Df} = \frac{\text{Drag force}}{\text{dynamic pressure} \quad \text{x wetted area}} = \frac{2R}{\rho u_0^2 \text{ x wetted area}} = \frac{2\tau_w}{\rho u_0^2}$$

Note that this is the same definition for the pipe friction coefficient C_f and it is in fact the same thing. It is used in the Darcy formula to calculate the pressure lost in pipes due to friction. For a smooth surface, it can be shown that $C_{Df} = 0.074\ (R_e)_x^{-1/5}$

$(Re)_l$ is the Reynolds number based on the length. $(R_e)_x = \frac{\rho u_o L}{\mu}$

<u>WORKED EXAMPLE No.1</u>

Calculate the drag force on each side of a thin smooth plate 2 m long and 1 m wide with the length parallel to a flow of fluid moving at 30 m/s. The density of the fluid is 800 kg/m^3 and the dynamic viscosity is 8 cP.

<u>SOLUTION</u>

$$(R_e)_x = \frac{\rho u_o L}{\mu} = \frac{800 \text{ x } 30 \text{ x } 2}{0.008} = 6 \text{ x } 10^6$$

$$C_{Df} = 0.074 \text{ x } (6 \text{ x} 10^6)^{-\frac{1}{5}} = 0.00326$$

$$\text{Dynamic pressure} = \frac{\rho u_0^2}{2} = \frac{800 \text{ x } 30^2}{2} = 360 \text{ kPa}$$

$$\tau_w = C_{Df} \text{ x dynamic pressure} = 0.00326 \text{ x } 360 \text{ x } 10^3 = 1173.6 \text{ Pa}$$

$$R = \tau_w \text{ x Wetted Area} = 1173.6 \text{ x } 2 \text{ x } 1 = 2347.2 \text{ N}$$

On a small area the drag is $dR = \tau_w\, dA$. If the body is not a thin plate and has an area inclined at an angle θ to the flow direction, the drag force in the direction of flow is $\tau_w\, dA \cos\theta$.

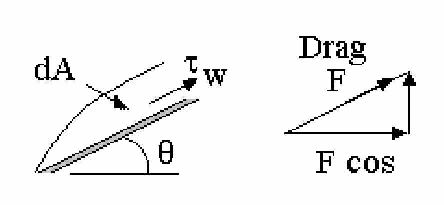


Fig.3

The drag force acting on the entire surface area is found by integrating over the entire area.

$$R = \oint \tau_w \cos\theta \, dA$$

Solving this equation requires more advanced studies concerning the boundary layer and students should refer to the classic textbooks on this subject.

SELF ASSESSMENT EXERCISE No.1

1. A smooth thin plate 5 m long and 1 m wide is placed in an air stream moving at 3 m/s with its length parallel with the flow. Calculate the drag force on each side of the plate. The density of the air is 1.2 kg/m^3 and the kinematic viscosity is 1.6 x 10^{-5} m^2/s. (0.128 N)

2. A pipe bore diameter D and length L has fully developed laminar flow throughout the entire length with a centre line velocity u_o. Given that the drag coefficient is given as $C_{Df} = 16/Re$ where $Re = \dfrac{\rho u_o D}{\mu}$, show that the drag force on the inside of the pipe is given as $R=8\pi\mu u_o L$ and hence the pressure loss in the pipe due to skin friction is $p_L = 32\mu u_o L/D^2$

1.2 FORM DRAG and WAKES

Form or pressure drag applies to bodies that are tall in comparison to the length in the direction of flow. Such bodies are called ***BLUFF BODIES***.

Consider the case below that could for example, be the pier of a bridge in a river. The water speeds up around the leading edges and the boundary layer quickly breaks away from the surface. Water is sucked in from behind the pier in the opposite direction.

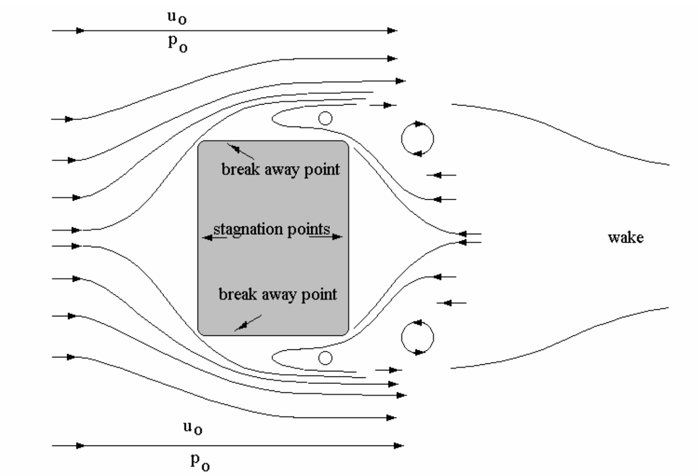


Fig. 4

The total effect is to produce eddy currents or whirl pools that are shed in the wake. There is a build up of positive pressure on the front and a negative pressure at the back. The pressure force resulting is the form drag. When the breakaway or separation point is at the front corner, the drag is almost entirely due to this effect but if the separation point moves along the side towards the back, then a boundary layer forms and skin friction drag is also produced. In reality, the drag is always a combination of skin friction and form drag. The degree of each depends upon the shape of the body.

The next diagram typifies what happens when fluid flows around a bluff object. The fluid speeds up around the front edge. Remember that the closer the streamlines, the faster the velocity. The line representing the maximum velocity is shown but also remember that this is the maximum at any point and this maximum value also increases as the stream lines get closer together.

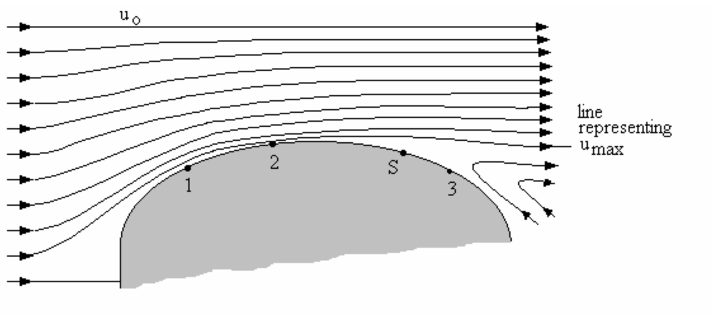


Fig. 5

Two important effects affect the drag.

Outside the boundary layer, the velocity increases up to point 2 so the pressure acting on the surface goes down. The boundary layer thickness δ gets smaller until at point S it is reduced to zero and the flow separates from the surface. At point 3, the pressure is negative. ***This change in pressure is responsible for the form drag.***

Inside the boundary layer, the velocity is reduced from u_{max} to zero and skin friction drag results.

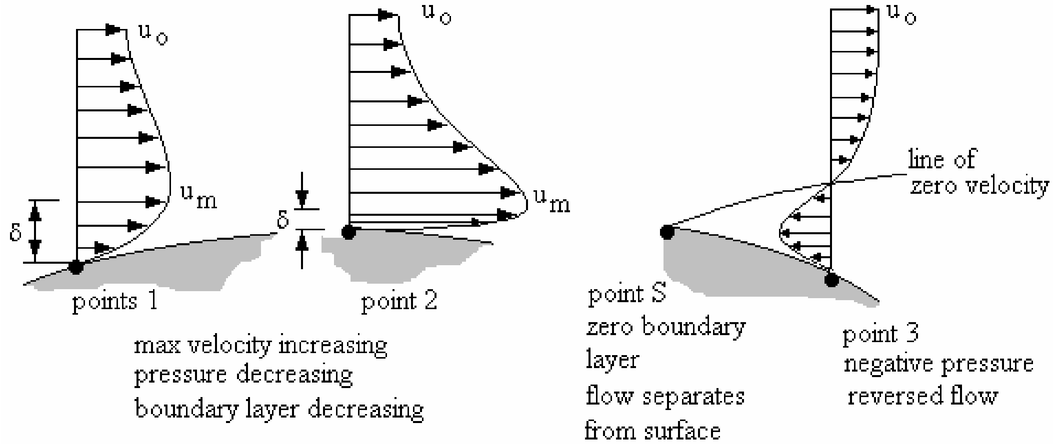


Fig. 6

In problems involving liquids with a free surface, a negative pressure shows up as a drop in level and the pressure build up on the front shows as a rise in level. If the object is totally immersed, the pressure on the front rises and a vacuum is formed at the back. This results in a pressure force opposing movement (form drag). The swirling flow forms ***vortices*** and the wake is an area of great turbulence behind the object that takes some distance to settle down and revert to the normal flow condition.

Here is an outline of the mathematical approach needed to solve the form drag.

Form drag is due to pressure changes only. The drag coefficient due to pressure only is denoted C_{Dp} and defined as

$$C_{Dp} = \frac{\text{Drag force}}{\text{dynamic pressure} \quad \text{x projected area}} = \frac{2R}{\rho u_0^2 \text{ x projected area}}$$

The projected area is the area of the outline of the shape projected at right angles to the flow. The pressure acting at any point on the surface is p. The force exerted by the pressure on a small surface area is p dA. If the surface is inclined at an angle θ to the general direction of flow, the force is p cosθ dA. The total force is found by integrating all over the surface.

$$R = \oint p\cos\theta \, dA$$

The pressure distribution over the surface is often expressed in the form of a pressure coefficient defined as follows.

$$C_p = \frac{2(p - p_o)}{\rho u_o^2}$$

p_o is the static pressure of the undisturbed fluid, u_o is the velocity of the undisturbed fluid and $\frac{\rho u_o^2}{2}$ is the dynamic pressure of the stream.

Consider any streamline that is affected by the surface. Applying Bernoulli between an undisturbed point and another point on the surface, we have the following.

$$p_o + \frac{\rho u_o^2}{2} = p + \frac{\rho u^2}{2} \qquad p - p_o = \frac{\rho}{2}(u_o^2 - u^2)$$

$$C_p = \frac{2(p - p_o)}{\rho u_o^2} = \frac{2\left(\frac{\rho}{2}(u_o^2 - u^2)\right)}{\rho u_o^2} = \frac{(u_o^2 - u^2)}{u_o^2} = 1 - \frac{u^2}{u_o^2}$$

In order to calculate the drag force, further knowledge about the velocity distribution over the object would be needed and students are again recommended to study the classic textbooks on this subject. The equation shows that if $u<u_o$ then the pressure is positive and if $u>u_o$ the pressure is negative.

1.3 TOTAL DRAG

It has been explained that a body usually experiences both skin friction drag and form drag. The total drag is the sum of both. This applies to aeroplanes and ships as well as bluff objects such as cylinders and spheres. The drag force on a body is very hard to predict by purely theoretical methods. Much of the data about drag forces is based on experimental data and the concept of a drag coefficient is widely used.

The ***DRAG COEFFICIENT*** is denoted $\mathbf{C_D}$ and is defined by the following expression.

$$C_D = \frac{\text{Resistance force}}{\text{Dynamic pressure x projected Area}} = \frac{2R}{\rho u_o^2 \text{ x projected Area}}$$

WORKED EXAMPLE No.2

A cylinder 80 mm diameter and 200 mm long is placed in a stream of fluid moving at 0.5 m/s. The axis of the cylinder is normal to the direction of flow. The density of the fluid is 800 kg/m^3. The drag force is measured and found to be 30 N.

Calculate the drag coefficient.

At a point on the surface, the pressure is measured as 96 Pa above ambient.

Calculate the velocity at this point.

SOLUTION

Projected area = 0.08 x 0.2 = 0.016 m^2

R = 30 N, u_o = 0.5 m/s ρ = 800 kg/m^3

Dynamic pressure = $\rho u^2/2$ = 800 x $0.5^2/2$ = 100 Pa

$$C_D = \frac{\text{Resistance force}}{\text{Dynamic pressure x projected Area}} = \frac{30}{100 \text{ x } 0.016} = 18.75$$

$$p - p_o = \frac{\rho}{2}(u_o^2 - u^2) \qquad 96 = \frac{800}{2}(0.5^2 - u^2) \qquad \frac{96 \text{ x } 2}{800} = (0.5^2 - u^2)$$

$$0.24 = 0.25 - u^2 \qquad u^2 = 0.01 \qquad u = 0.1 \text{ m/s}$$

1.4 APPLICATION TO A CYLINDER

The drag coefficient is defined as : $C_D = \dfrac{2R}{\rho u_o^2 \text{ x projected Area}}$ The projected Area is LD where L is the length and D the diameter. The drag around long cylinders is more predictable than for short cylinders and the following applies to long cylinders. Much research has been carried out into the relationship between drag and Reynolds number. $Re = \dfrac{\rho u_o d}{\mu}$ and d is the diameter of the cylinder.

At very small velocities, (Re <0.5) the fluid sticks to the cylinder all the way round and never separates from the cylinder. This produces a streamline pattern similar to that of an ideal fluid. The drag coefficient is at its highest and is mainly due to skin friction. The pressure distribution shows that the dynamic pressure is achieved at the front stagnation point and vacuum equal to three dynamic pressures exists at the top and bottom where the velocity is at its greatest.

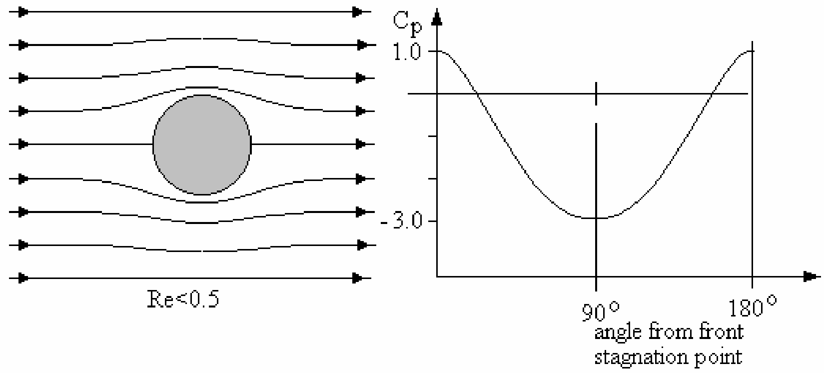


Fig. 7

As the velocity increases the boundary layer breaks away and eddies are formed behind. The drag becomes increasingly due to the pressure build up at the front and pressure drop at the back.

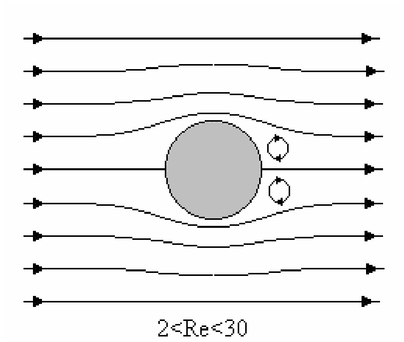


Fig. 8

Further increases in the velocity cause the eddies to elongate and the drag coefficient becomes nearly constant. The pressure distribution shows that ambient pressure exists at the rear of the cylinder.

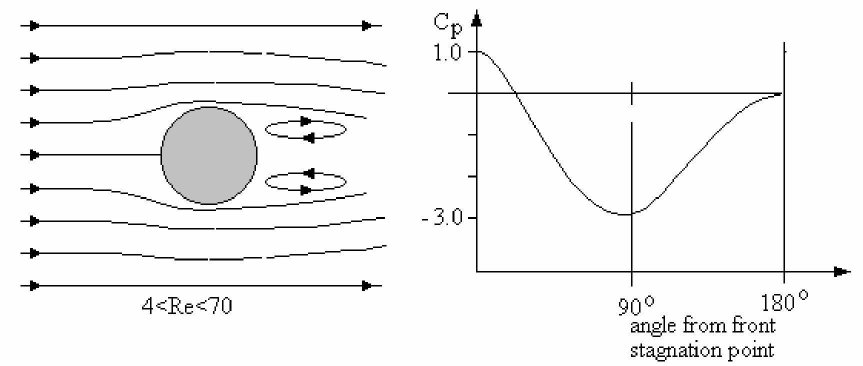


Fig. 9

At a Reynolds number of around 90 the vortices break away alternatively from the top and bottom of the cylinder producing a vortex street in the wake. The pressure distribution shows a vacuum at the rear.

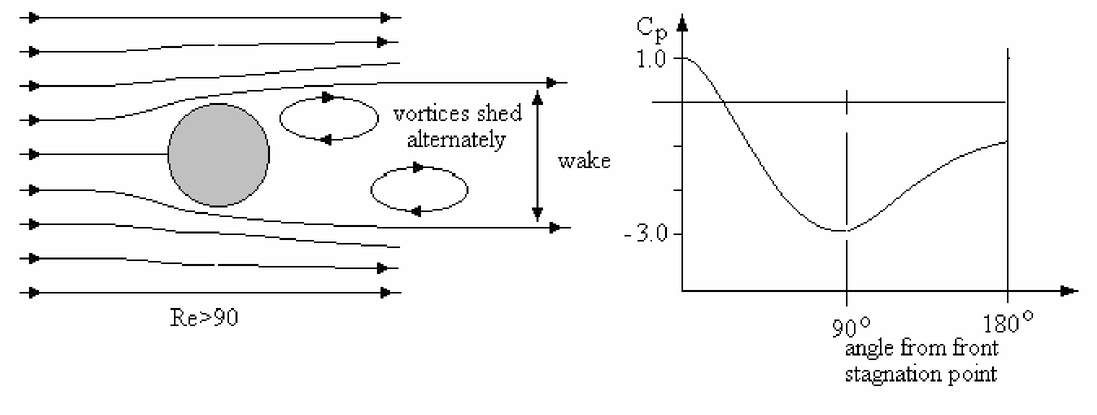


Fig.10

Up to a Reynolds number of about 2 x 10^5, the drag coefficient is constant with a value of approximately 1. The drag is now almost entirely due to pressure. Up to this velocity, the boundary layer has remained laminar but at higher velocities, flow within the boundary layer becomes turbulent. The point of separation moves back producing a narrow wake and a pronounced drop in the drag coefficient.

When the wake contains vortices shed alternately from the top and bottom, they produce alternating forces on the structure. If the structure resonates with the frequency of the vortex shedding, it may oscillate and produce catastrophic damage. This is a problem with tall chimneys and suspension bridges. The vortex shedding may produce audible sound.

Fig. 12 shows an approximate relationship between C_D and R_e for a cylinder and a sphere.

SELF ASSESSMENT EXERCISE No.2

1. Calculate the drag force for a cylindrical chimney 0.9 m diameter and 50 m tall in a wind blowing at 30 m/s given that the drag coefficient is 0.8. The density of the air is 1.2 kg/m^3. (19.44 N)

2. Using the graph (fig.12) to find the drag coefficient, determine the drag force per metre length acting on an overhead power line 30 mm diameter when the wind blows at 8 m/s. The density of air may be taken as 1.25 kg/m^3 and the kinematic viscosity as 1.5 x 10^{-5} m^2/s. (1.8 N).

1.5 APPLICATION TO SPHERES

The relationship between drag and Reynolds number is roughly the same as for a cylinder but it is more predictable. The Reynolds number is $\mathrm{Re} = \frac{\rho u_o d}{\mu}$ where d is the diameter of the sphere. The projected area of a sphere of diameter d is ¼ πd^2. In this case, the expression for the drag coefficient is as follows. $C_D = \frac{8R}{\rho u^2 \text{ x } \pi d^2}$.

At very small Reynolds numbers (less than 0.2) the flow stays attached to the sphere all the way around and this is called Stokes flow. The drag is at its highest in this region.

As the velocity increases, the boundary layer separates at the rear stagnation point and moves forward. A toroidal vortex is formed. For 0.2<Re<500 the flow is called Allen flow.

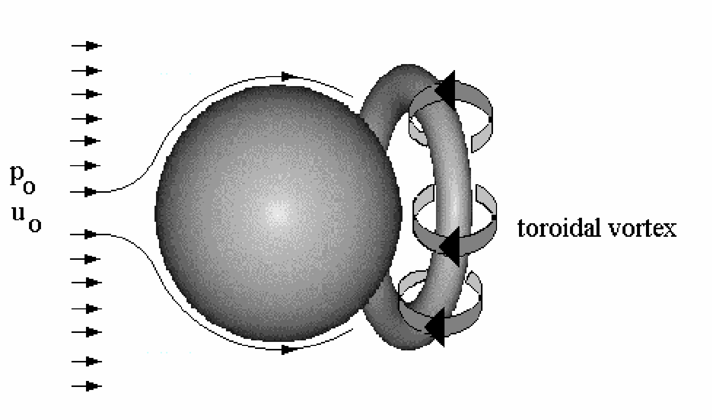


Fig.11

The breakaway or separation point reaches a stable position approximately 80° from the front stagnation point and this happens when R_e is about 1000. For 500<R_e the flow is called Newton flow. The drag coefficient remains constant at about 0.4. Depending on various factors, when R_e reaches 10^5 or larger, the boundary layer becomes totally turbulent and the separation point moves back again forming a smaller wake and a sudden drop in the drag coefficient to about 0.3. An empirical formula that covers the range $0.2 < R_e < 10^5$ is as follows.

$$C_D = \frac{24}{R_e} + \frac{6}{1+\sqrt{R_e}} + 0.4$$

Fig. 12 shows this approximate relationship between C_D and Re.

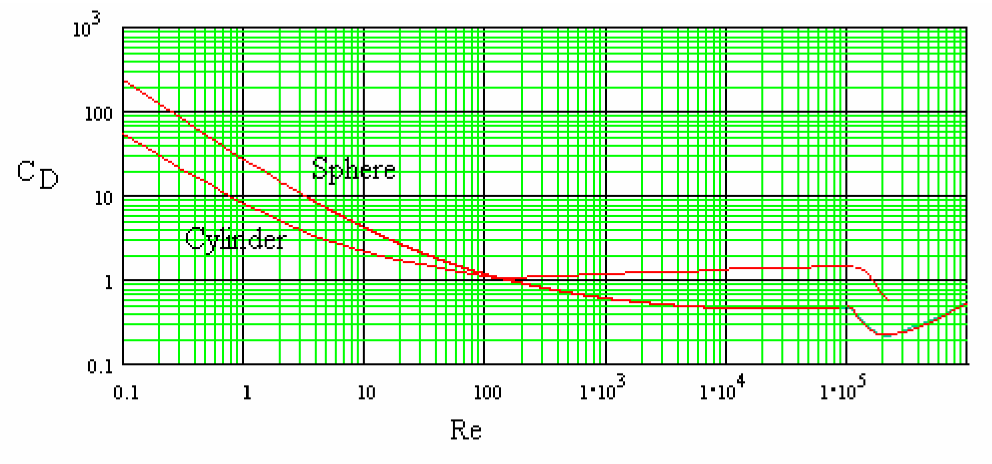


Fig.12

WORKED EXAMPLE No.3

A sphere diameter 40 mm moves through a fluid of density 750 kg/m³ and dynamic viscosity 50 cP with a velocity of 0.6 m/s. Note 1 cP = 0.001 Ns/m².

Calculate the drag on the sphere.

Use the graph to obtain the drag coefficient.

SOLUTION

$$\mathrm{Re} = \frac{\rho u d}{\mu} = \frac{750 \text{ x } 0.6 \text{ x } 0.04}{0.05} = 360$$

from the graph $C_D = 0.8$

$$C_D = \frac{2R}{\rho u^2 \text{ x projected Area}} \qquad \text{Projected area} = \pi\frac{d^2}{4} = \pi\frac{0.04^2}{4} = 1.2566 \text{ x } 10^{-3}\,\mathrm{m}^2$$

$$R = \frac{C_D \rho u^2 A}{2} = \frac{0.8 \text{ x } 750 \text{ x } 0.6^2 \text{ x } 1.2566 \text{ x } 10^{-3}}{2} = 0.136 \text{ N}$$

1.6 TERMINAL VELOCITY

When a body falls under the action of gravity, a point is reached, where the drag force is equal and opposite to the force of gravity. When this condition is reached, the body stops accelerating and the terminal velocity reached. Small particles settling in a liquid are usually modelled as small spheres and the preceding work may be used to calculate the terminal velocity of small bodies settling in a liquid. A good application of this is the falling sphere viscometer described in chapter one.

For a body immersed in a liquid, the buoyant weight is W and this is equal to the viscous resistance R when the terminal velocity is reached.

R = W = volume x gravity x density difference $= \dfrac{\pi d^3 g(\rho_s - \rho_f)}{6}$

ρ_s = density of the sphere material
ρ_f = density of fluid
d = sphere diameter

STOKES' FLOW

For $R_e<0.2$ the flow is called Stokes flow and Stokes showed that $R = 3\pi d\mu u_t$
For a falling sphere viscometer, Stokes flow applies. Equating the drag force and the buoyant weight we get

$$3\pi\pi d\mu_t = \frac{\pi d^3 g(\rho_s - \rho_f)}{6}$$

$$\mu = \frac{d^2 g(\rho_s - \rho_f)}{18u_t}$$

The terminal velocity for Stokes flow is $u_t = \dfrac{d^2 g(\rho_s - \rho_f)}{18\mu}$

This formula assumes a fluid of infinite width but in a falling sphere viscometer, the liquid is squeezed between the sphere and the tube walls and additional viscous resistance is produced. The Faxen correction factor F is used to correct the result.

WORKED EXAMPLE No.4

The terminal velocity of a steel sphere falling in a liquid is 0.03 m/s. The sphere is 1 mm diameter and the density of the steel is 7830 kg/m^3. The density of the liquid is 800 kg/m^3. Calculate the dynamic and kinematic viscosity of the liquid.

SOLUTION

Assuming Stokes' flow the viscosity is found from the following equation.

$$\mu = \frac{d^2 g(\rho_s - \rho_f)}{18u_t} = \frac{0.001^2 \text{ x } 9.81 \text{ x } (7830 - 800)}{18 \text{ x } 0.03} = 0.1277 \text{ Ns/m}^2 = 127.7 \text{ cP}$$

$$\nu = \frac{\mu}{\rho_s} = \frac{0.1277}{800} = 0.0001596 \text{ m}^2\text{/s} = 159.6 \text{ cSt}$$

Check the Reynolds number. $R_e = \dfrac{\rho_f ud}{\mu} = \dfrac{800 \text{ x } 0.03 \text{ x } 0.001}{0.0547} = 0.188$

As this is smaller than 0.2 the assumption of Stokes' flow is correct.

ALLEN FLOW

For $0.2 < R_e < 500$ the flow is called Allen flow and the following expression gives the empirical relationship between drag and Reynolds number. $\mathbf{C_D=18.5R_e^{-0.6}}$

Equating for C_D gives the following result. $C_D = \dfrac{8R}{\rho_f u_t^2\ \pi\ {}^{2}} = 18.5R_e^{-0.6}$

Substitute $R = \dfrac{\pi d^3 g(\rho_s - \rho_f)}{6}$

$$C_D = \frac{8dg(\rho_s - \rho_f)}{6\rho_f u_t^2} = 18.5R_e^{-0.6} = 18.5\left(\frac{\rho_f u_t d}{\mu}\right)^{-0.6}$$

$$\frac{8dg(\rho_s - \rho_f)}{6\rho_f u_t^2} = 18.5\left(\frac{\rho_f u_t d}{\mu}\right)^{-0.6}$$

From this equation the velocity u_t may be found.

NEWTON FLOW

For $500 < R_e < 10^5$ C_D takes on a constant value of 0.44.

Equating for C_D gives the following. $C_D = \dfrac{8R}{\rho_f u_t^2\ \pi\ {}^{2}} = 0.44$

Substitute $R = \dfrac{\pi d^3 g(\rho_s - \rho_f)}{6}$

$$\frac{8dg(\rho_s - \rho_f)}{6\rho_f u_t^2} = 0.44 \qquad u_t = \sqrt{\frac{29.73dg(\rho_s - \rho_f)}{\rho_f}}$$

When solving the terminal velocity, you should always check the value of the Reynolds number to see if the criterion used is valid.

WORKED EXAMPLE No.5

Small glass spheres are suspended in an up wards flow of water moving with a mean velocity of 1 m/s. Calculate the diameter of the spheres. The density of glass is 2630 kg/m^3. The density of water is 1000 kg/m^3 and the dynamic viscosity is 1 cP.

SOLUTION

First, try the Newton flow equation. This is the easiest.

$$u_t = \sqrt{\frac{29.73d\,g(\rho_s - \rho_f)}{\rho_f}}$$

$$d = \frac{u_t^2\rho_f}{29.73\,g(\rho_s - \rho_f)} = \frac{1^2 \text{ x } 1000}{29.73 \text{ x } 9.81 \text{ x } (2630 - 1000)} = 0.0021\,\text{m or } 2.1\,\text{mm}$$

Check the Reynolds number.

$$R_e = \frac{\rho_f u_t d}{\mu} = \frac{1000 \text{ x } 1 \text{ x } 0.0021}{0.001} = 2103$$

The assumption of Newton flow was correct so the answer is valid.

WORKED EXAMPLE No.6

Repeat the last question but this time with a velocity of 0.05 m/s. Determine the type of flow that exists.

SOLUTION

If no assumptions are made, we should use the general formula $C_D = \frac{24}{R_e} + \frac{6}{1+\sqrt{R_e}} + 0.4$

$$R_e = \frac{\rho_f u_t d}{\mu} = \frac{1000 \text{ x } 0.05 \text{ x d}}{0.001} = 50\,000d$$

$$C_D = \frac{24}{R_e} + \frac{6}{1+\sqrt{R_e}} + 0.4 = \frac{24}{50000d} + \frac{6}{1+\sqrt{50000d}} + 0.4$$

$$C_D = 0.00048d^{-1} + \frac{6}{1+223.6d^{0.5}} + 0.4$$

$$C_D = \frac{8dg(\rho_s - \rho_f)}{6\rho_f u^2} = \frac{8d \text{ x } 9.81 \text{ x } (2630-1000)}{6 \text{ x } 1000 \text{ x } 0.05^2} = 8528.16d$$

$$8528.16d = 0.00048d^{-1} + \frac{6}{1+223.6d^{0.5}} + 0.4$$

This should be solved by any method known to you such as plotting two functions and finding the point of interception.

$$f1(d) = 8528.16d$$

$$f2(d) = 0.00048d^{-1} + \frac{6}{1+223.6d^{0.5}} + 0.4$$

The graph below gives an answer of d = 0.35 mm.

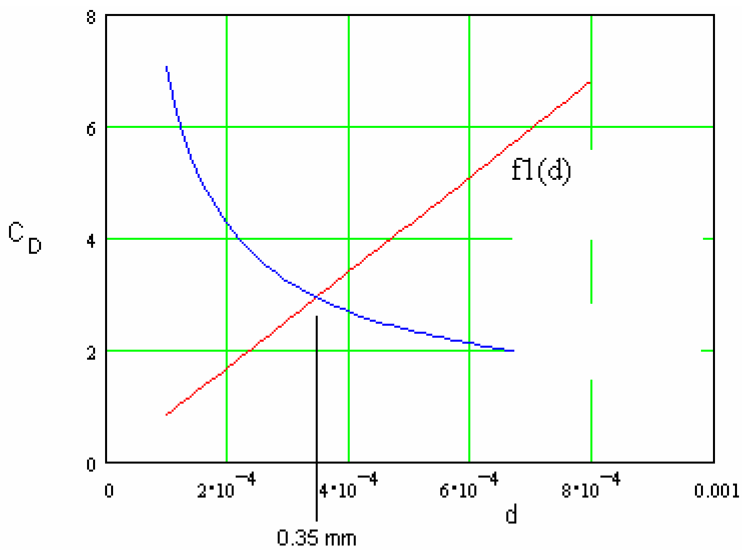


Fig. 13

Checking the Reynolds' number $R_e = \frac{\rho_f u_t d}{\mu} = \frac{1000 \text{ x } 0.05 \text{ x } 0.00035}{0.001} = 17.5$

This puts the flow in the Allen's flow section.

SELF ASSESSMENT EXERCISE No.3

1. a. Explain the term Stokes flow and terminal velocity.

b. Show that the terminal velocity of a spherical particle with Stokes flow is given by the formula
$u = d^2g(\rho_s - \rho_f)/18\mu$

Go on to show that $C_D=24/R_e$

2. Calculate the largest diameter sphere that can be lifted upwards by a vertical flow of water moving at 1 m/s. The sphere is made of glass with a density of 2630 kg/m^3. The water has a density of 998 kg/m^3 and a dynamic viscosity of 1 cP. (20.7 mm)

3. Using the same data for the sphere and water as in Q2, calculate the diameter of the largest sphere that can be lifted upwards by a vertical flow of water moving at 0.5 m/s. (5.95 mm).

4. Using the graph (fig. 12) to obtain the drag coefficient of a sphere, determine the drag on a totally immersed sphere 0.2 m diameter moving at 0.3 m/s in sea water. The density of the water is 1025 kg/m^3 and the dynamic viscosity is 1.05×10^{-3} Ns/m^2. (0.639 N).

5. A glass sphere of diameter 1.5 mm and density 2 500 kg/m^3 is allowed to fall through water under the action of gravity. The density of the water is 1000 kg/m^3 and the dynamic viscosity is 1 cP.

Calculate the terminal velocity assuming the drag coefficient is
$C_D = 24\, R_e^{-1}\,(1+ 0.15R_e^{\,0.687})$ (Ans. 0.215 m/s

EDEXCEL HIGHER

FLUID MECHANICS H1 UNIT 8

NQF LEVEL 4

OUTCOME 3 - THE FLOW OF REAL FLUIDS

TUTORIAL 5 – DIMENSIONAL ANALYSIS

3 **Flow of real fluids**

Head losses**:** head loss in pipes by Darcy's formula; Moody diagram; head loss due to sudden enlargement and contraction of pipe diameter; head loss at entrance to a pipe; head loss in valves; flow between reservoirs due to gravity; hydraulic gradient; siphons; hammer blow in pipes

Reynolds' number**:** inertia and viscous resistance forces; laminar and turbulent flow; critical velocities

Viscous drag**:** dynamic pressure; form drag; skin friction drag; drag coefficient

Dimensional analysis**:** checking validity of equations such as those for pressure at depth; thrust on immersed surfaces and impact of a jet; forecasting the form of possible equations such as those for Darcy's formula and critical velocity in pipes

In this section you will do the following.

- Learn the basic system of dimensions.
- Find the relationship between variables affecting a phenomenon.
- Define and use dimensionless numbers.

5.1 BASIC DIMENSIONS

All quantities used in engineering can be reduced to six basic dimensions. These are the dimensions of

Mass	M
Length	L
Time	T
Temperature	θ
Electric Current	I
Luminous Intensity	J

The last two are not used in fluid mechanics and temperature is only used sometimes.

All engineering quantities can be defined in terms of the four basic dimensions M,L,T and θ. We could use the S.I. units of kilogrammes, metres, seconds and Kelvins, or any other system of units, but if we stick to M,L,T and θ we free ourselves of any constraints to a particular system of measurements.

Let's now explain the above with an example. Consider the quantity *density*. The S.I. units are kg/m^3 and the imperial units are lb/in^3. In our system the units would be $Mass/Length^3$ or M/L^3. It will be easier in the work ahead if we revert to the inverse indice notation and write it as ML^{-3}.

Other engineering quantities need a little more thought when writing out the basic MLTθ dimensions. The most important of these is the unit of force or the Newton in the S.I. system. Engineers have opted to define force as that which is needed to accelerate a mass such that 1 N is needed to accelerate 1 kg at 1 m/s^2. From this we find that the Newton is a derived unit equal to 1 kg m/s^2. In our system the dimensions of force become MLT^{-2}. This must be considered when writing down the dimensions of anything containing force.

Another unit that produces problems is that of angle. Angle is a ratio of two sides of a triangle and so has no units nor dimensions at all. This also applies to revolutions which is an angular measurement. Strain is also a ratio and has no units nor dimensions. Angle and strain are in fact examples of dimensionless quantities which will be considered in detail later.

WORKED EXAMPLE No.1

Write down the basic dimensions of pressure p.

SOLUTION

Pressure is defined as p = Force/Area

The S.I. unit of pressure is the Pascal which is the name for $1N/m^2$.
Since force is MLT^{-2} and area is L^2 then the basic dimensions of pressure are

$ML^{-1}T^{-2}$

When solving problems it is useful to use a notation to indicate the MLT dimensions of a quantity and in this case we would write

$$[p] = ML^{-1}T^{-2}$$

WORKED EXAMPLE No. 2

Deduce the basic dimensions of dynamic viscosity.

SOLUTION

Dynamic viscosity was defined in an earlier tutorial from the formula $\tau = \mu du/dy$

τ is the shear stress, du/dy is the velocity gradient and μ is the dynamic viscosity. From this we have $\mu = \tau\, dy/du$

Shear stress is force/area.
The basic dimensions of force are MLT^{-2}
The basic dimensions of area are L^2.
The basic dimensions of shear stress are $ML^{-1}T^{-2}$.
The basic dimensions of distance y are L.
The basic dimensions of velocity v are LT^{-1}.
It follows that the basic dimension of dy/du (a differential coefficient) is T.
The basic dimensions of dynamic viscosity are hence $(ML^{-1}T^{-2})(T) = ML^{-1}T^{-1}$.

$[\mu] = ML^{-1}T^{-1}$.

5.2 LIST OF QUANTITIES AND DIMENSIONS FOR REFERENCE.

AREA	$(LENGTH)^2$	L^2
VOLUME	$(LENGTH)^3$	L^3
VELOCITY	LENGTH/TIME	LT^{-1}
ACCELERATION	$LENGTH/(TIME^2)$	LT^{-2}
ROTATIONAL SPEED	REVOLUTIONS/TIME	T^{-1}
FREQUENCY	CYCLES/TIME	T^{-1}
ANGULAR VELOCITY	ANGLE/TIME	T^{-1}
ANGULAR ACCELERATION	$ANGLE/(TIME)^2$	T^{-2}
FORCE	MASS X ACCELERATION	MLT^{-2}
ENERGY	FORCE X DISTANCE	ML^2T^{-2}
POWER	ENERGY/TIME	ML^2T^{-3}
DENSITY	MASS/VOLUME	ML^{-3}
DYNAMIC VISCOSITY	STRESS/VELOCITY GRADIENT	$ML^{-1}T^{-1}$
KINEMATIC VISCOSITY	DYNAMIC VIS/DENSITY	L^2T^{-1}
PRESSURE	FORCE/AREA	$ML^{-1}T^{-2}$
SPECIFIC HEAT CAPACITY	ENERGY/(MASS X TEMP)	$L^2T^{-2}\theta^{-1}$
TORQUE	FORCE X LENGTH	ML^2T^{-2}
BULK MODULUS	PRESSURE/STRAIN	$ML^{-1}T^{-2}$

5.3 HOMOGENEOUS EQUATIONS

All equations must be homogeneous.
Consider the equation $F = 3 + T/R$
F is force, T is torque and R is radius.
Rearranging we have $3 = F - T/R$

Examine the units.

F is Newton. T is Newton metre and R is metre.

hence $3 = F\ (N) - T/R\ (N\ m)/m$
$3 = F(N) - T/R\ (N)$

It follows that the number 3 must represent 3 Newton. It also follows that the unit of F and T/R must both be Newton. If this was not so, the equation would be nonsense. In other words all the components of an equation which add together must have the same units. You cannot add dissimilar quantities. For example you cannot say that 5 apples + 6 pears = 11 plums. This is clearly nonsense. When all parts of an equation that add together have the same dimensions, then the equation is homogeneous.

WORKED EXAMPLE No. 3

Show that the equation Power = Force x velocity is homogeneous in both S.I. units and basic dimensions.

SOLUTION

The equation to be checked is $P = F\ v$

The S.I. Unit of power (P) is the Watt. The Watt is a Joule per second. A Joule is a Newton metre of energy. Hence a Watt is 1 N m/s.

The S.I. unit of force (F) is the Newton and of velocity (v) is the metre/second.

The units of F v are hence N m/s.

It follows that both sides of the equation have S.I. units of N m/s so the equation is homogeneous.

Writing out the MLT dimensions of each term we have

$[P] = ML^2T^{-3}$
$[v] = LT^{-1}$
$[F] = MLT^{-2}$

Substituting into the equation we have $ML^2T^{-3} = MLT^{-2}\ LT^{-1} = ML^2T^{-3}$

Hence the equation is homogeneous.

5.4 **INDECIAL EQUATIONS**

When a phenomenon occurs, such as a swinging pendulum as shown in figure 14 we observe the variables that effect each other. In this case we observe that the frequency, (f) of the pendulum is affected by the length (l) and the value of gravity (g). We may say that frequency is a function of l and g. In equation form this is as follows.

$f = \phi(l,g)$ where ϕ is the function sign.

When we remove the function sign we must put in a constant because there is an unknown number and we must allocate unknown indices to l and g because we do know not what if any they are. The equation is written as follows. $f = C\, l^a\, g^b$

C is a constant and has no units. a and b are unknown indices.

This form of relating variables is called an indicial equation. The important point here is that because we know the units or dimensions of all the variables, we can solve the unknown indices.

WORKED EXAMPLE No. 4

Solve the relationship between f, l and g for the simple pendulum.

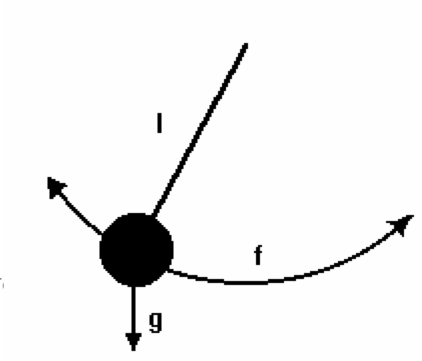


Fig.1

SOLUTION

First write down the indecial form of the equation (covered ov

$f = C\, l^a\, g^b$

Next write down the basic dimensions of all the variables.

$[f] = T^{-1}$

$[l] = L^1$

$[g] = LT^{-2}$

Next substitute the dimensions in place of the variables.

$T^{-1} = (L^1)^a\, (LT^{-2})^b$

Next tidy up the equation. $T^{-1} = L^{1a}\, L^b T^{-2b}$

Since the equation must be homogeneous then the power of each dimension must be the same on the left and right side of the equation. If a dimension does not appear at all then it is implied that it exists to the power of zero. We may write them in until we get use to it. The equation is written as follows.

$M^0 L^0 T^{-1} = L^{1a}\, LT^{-2b} M^0$

Next we equate powers of each dimension. First equate powers of Time.

$T^{-1} = T^{-2b}$ $\quad -1 = -2b \quad$ $b = 1/2$

Next equate powers of Length.

$L^0 = L^{1a}\, L^b$ $\quad 0 = 1a + b \quad$ hence $a = -b = -1/2$

$M^0 = M^0$ yields nothing in this case.

Now substitute the values of a and b back into the original equation and we have the following.

$f = C\, l^{-½}\, g^{½}$ $\quad f = C\,(g/l)^{½}$

The frequency of a pendulum may be derived from basic mechanics and shown to be

$f = (1/2\pi)(g/l)^{½}$

If we did not know how to find $C = (½\,\pi)$ from basic mechanics, then we know that if we conducted an experiment and measured the values f for various values of l and g, we could find C by plotting a graph of f against $(g/l)^{½}$. This is the importance of dimensional analysis to fluid mechanics. We are able to determine the basic relationships and then conduct experiments and determine the remaining unknown constants. We are able to plot graphs because we know what to plot against what.

SELF ASSESSMENT EXERCISE No. 1

1. It is observed that the velocity 'v' of a liquid leaving a nozzle depends upon the pressure drop 'p' and the density 'ρ'. Show that the relationship between them is of the form

$$v = C\left(\frac{p}{\rho}\right)^{\frac{1}{2}}$$

2. It is observed that the speed of a sound in 'a' in a liquid depends upon the density 'ρ' and the bulk modulus 'K'. Show that the relationship between them is

$$a = C\left(\frac{K}{\rho}\right)^{\frac{1}{2}}$$

3. It is observed that the frequency of oscillation of a guitar string 'f' depends upon the mass 'm', the length 'l' and tension 'F'. Show that the relationship between them is $f = C\left(\frac{F}{ml}\right)^{\frac{1}{2}}$

5.5 DIMENSIONLESS NUMBERS

We will now consider cases where the number of unknown indices to be solved, exceed the number of equations to solve them. This leads into the use of dimensionless numbers.

Consider that typically a problem uses only the three dimensions M, L and T. This will yield 3 simultaneous equations in the solution. If the number of variables in the equation gives 4 indices say a, b, c and d, then one of them cannot be resolved and the others may only be found in terms of it.

In general there are n unknown indices and m variables. There will be m-n unknown indices. This is best shown through a worked example.

WORKED EXAMPLE No. 5

The pressure drop per unit length 'p' due to friction in a pipe depends upon the diameter 'D' , the mean velocity 'v' , the density 'ρ' and the dynamic viscosity 'μ'. Find the relationship between these variables.

SOLUTION

$p = \text{function}(D\ v\ \rho\ \mu) = K\ D^a\ v^b\ \rho^c\ \mu^d$

p is pressure per metre

$[p] = ML^{-2}T^{-2}$

$[D] = L$

$[v] = LT^{-1}$

$[\rho] = ML^{-3}$

$[\mu] = ML^{-1}T^{-1}$

$ML^{-2}T^{-2} = L^a (LT^{-1})^b (ML^{-3})^c (ML^{-1}T^{-1})^d$

$ML^{-2}T^{-2} = L^{a+b-3c-d} M^{c+d} T^{-b-d}$

The problem is now deciding which index not to solve. The best way is to use experience gained from doing problems. Viscosity is the quantity that causes viscous friction so the index associated with it (d) is the one to identify. We will resolve a, b and c in terms of d.

TIME $-2 = -b - d$ hence $b = 2 - d$ is as far as we can resolve

MASS $1 = c + d$ hence $c = 1 - d$
LENGTH $-2 = a + b - 3c - d$

$-2 = a + (2 - d) - 3(1-d) - d$ hence $a = -1 - d$

Now put these back into the original formula.

$p = K D^{-1-d} v^{2-d} \rho^{1-d} \mu^d$

Now group the quantities with same power together as follows :

$p = K\{\rho v^2 D^{-1}\} \{\mu \rho^{-1} v^{-1} D^{-1}\}^d$

Remember that p was pressure drop per unit length so the pressure loss over a length L is
$P = K L\{\rho v^2 D^{-1}\} \{\mu \rho^{-1} v^{-1} D^{-1}\}^d$

We have two unknown constants K and d. The usefulness of dimensional analysis is that it tells us the form of the equation so we can deduce how to present experimental data. With suitable experiments we could now find K and d.

Note that this equation matches up with Poiseuille's equation which gives the relationship as :

$p = 32 \mu L v D^{-2}$

from which it may be deduced that $K = 32$ and $d = 1$ (laminar flow only)

The term $\{\rho v D \mu^{-1}\}$ has no units. If you check it out all the units will cancel. This is a ***DIMENSIONLESS NUMBER***, and it is named after Reynolds.

Reynolds Number is denoted R_e. The whole equation can be put into a dimensionless form as follows.

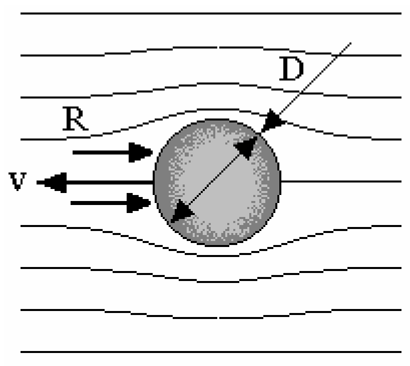
$$\{p \rho^{-1} L^{-1} v^{-2} D^1\} = K \{\mu \rho^{-1} v^{-1} D^{-1}\}^d$$

$$\{p \rho^{-1} L^{-1} v^{-2} D^1\} = \text{function}(R_e)$$

This is a dimensionless equation. The term $\{p\rho^{-1} L^{-1} v^{-2} D^1\}$ is also a dimensionless number.

Let us now examine another similar problem.

WORKED EXAMPLE No. 6



Consider a sphere moving through an viscous fluid Completely submerged. The resistance to motion R depends upon the diameter D, the velocity v, the density ρ and the dynamic viscosity μ.
Find the equation that relates the variables.

Figure 2

$R = \text{function}(D\ v\ \rho\ \mu) = K\ D^a\ v^b\ \rho^c\ \mu^d$

First write out the MLT dimensions.

$[R] = ML^1T^{-2}$

$[D] = L \qquad ML^1T^{-2} = L^a (LT^{-1})^b\ (ML^{-3})^c (ML^{-1}T^{-1})^d$

$[v] = LT^{-1} \qquad ML^1T^{-2} = L^{a+b-3c-d}\ M^{c+d}\ T^{-b-d}$

$[\rho] = ML^{-3}$

$[\mu] = ML^{-1}T^{-1}$

Viscosity is the quantity which causes viscous friction so the index associated with it (d) is the one to identify. We will resolve a,b and c in terms of d as before.

TIME $-2 = -b - d$ hence $b = 2 - d$ is as far as we can resolve b
MASS $1 = c + d$ hence $c = 1 - d$
LENGTH $1 = a + b - 3c - d$
$1 = a + (2 - d) - 3(1 - d) - d$ hence $a = 2 - d$

Now put these back into the original formula.

$$R = K\ D^{2-d}\ v^{2-d}\ \rho^{1-d}\ \mu^d$$

Now group the quantities with same power together as follows :

$$R = K\{\rho v^2 D^2\}\ \{\mu\ \rho^{-1} v^{-1} D^{-1}\}^d$$

$$R\{\rho v^2 D^2\}^{-1} = K\ \{\mu\ \rho^{-1} v^{-1} D^{-1}\}^d$$

The term $\{\rho v D \mu^{-1}\}$ is the Reynolds Number R_e and the term $R\{\rho v^2 D^2\}^{-1}$ is called the Newton Number N_e. Hence the relationship between the variables may be written as follows.

$R\{\rho v^2 D^2\}^{-1} = \text{function}\ \{\rho v D\ \mu^{-1}\}$

$N_e = \text{function}\ (R_e)$

Once the basic relationship between the variables has been determined, experiments can be conducted to find the parameters in the equation. For the case of the sphere in an incompressible fluid we have shown that

$N_e = \text{function}\ (R_e)$ Or put another way $N_e = K\ (R_e)^n$

K is a constant of proportionality and n is an unknown index (equivalent to -d in the earlier lines). In logarithmic form the equation is

$\log(N_e) = \log(K) + n \log(R_e)$

This is a straight line graph from which log K and n are taken. Without dimensional analysis we would not have known how to present the information and plot it. The procedure now would be to conduct an experiment and plot log(Ne) against log(Re). From the graph we would then determine K and n.

5.6 BUCKINGHAM'S Π (Pi) THEORY

Many people prefer to find the dimensionless numbers by intuitive methods. Buckingham's theory is based on the knowledge that if there are m basic dimensions and n variables, then there are m - n dimensionless numbers. Consider worked example No.12 again. We had the basic equation

$$R = \text{function}(D\ v\ \rho\ \mu)$$

There are 5 quantities and there will be 3 basic dimensions ML and T. This means that there will be 2 dimensionless numbers Π_1 and Π_2. These numbers are found by choosing two prime quantities (R and μ).

Π_1 is the group formed between μ and D v ρ

Π_2 is the group formed between R and D v ρ

First taking μ. Experience tells us that this will be the Reynolds number but suppose we don't know this.

The dimensions of μ are $ML^{-1}T^{-1}$

The dimensions of D v ρ must be arranged to be the same.

$$\mu = \Pi_1\ D^a v^b\ \rho^c$$

$$M^1L^{-1}T^{-1} = \Pi_1\ (L)^a\ (LT^{-1})^b\ (ML^{-3})^c$$

Time	**-1 = -b**	**b = 1**
Mass		**c = 1**
Length	-1 = a + b -3c	
	-1 = a + 1 – 3	a=1

$$\mu = \Pi_1\ D^1 v^1\ \rho^1$$

$$\Pi_1 = \frac{\mu}{Dv\rho}$$

The second number must be formed by combining R with ρ,v and D

$$R = \Pi_2\ D^a v^b\ \rho^c$$

$$MLT^{-2} = \Pi_2\ (L)^a\ (LT^{-1})^b (ML^{-3})^c$$

Time	**-2 = -b**	**b = 2**
Mass		**c = 1**
Length	1 = a + b -3c	
	1 = a + 2 – 3	a = 2

$$R = \Pi_2\ D^2 v^2\ \rho^1$$

$$\Pi_2 = \frac{R}{\rho v^2 D^2}$$

The dimensionless equation is **$\Pi_2 = f(\Pi_1)$**

WORKED EXAMPLE No. 7

The resistance to motion 'R' for a sphere of diameter 'D' moving at constant velocity 'v' through a compressible fluid is dependant upon the density 'ρ' and the bulk modulus 'K'. The resistance is primarily due to the compression of the fluid in front of the sphere. Show that the dimensionless relationship between these quantities is N_e = function (M_a)

SOLUTION

R = function (D v ρ K) = $C\ D^a\ v^b\ \rho^c\ K^d$

There are 3 dimensions and 5 quantities so there will be 5 –3 = 2 dimensionless numbers. Identify that the one dimensionless group will be formed with R and the other with K.

Π_1 is the group formed between K and D v ρ

Π_2 is the group formed between R and D v ρ

$K = \Pi_2\, D^a\, v^b\, \rho^c$

$[K] = ML^{-1}T^{-2}$
$[D] = L$
$[v] = LT^{-1}$
$[\rho] = ML^{-3}$

$ML^{-1}T^{-2} = L^a (LT^{-1})^b (ML^{-3})^c$

$ML^{-1}T^{-2} = L^{a+b-3c}M^c T^{-b}$

Time **-2 = -b** **b = 2**
Mass **c = 1**
Length -1 = a + b -3c
-1 = a + 2 – 3 a = 0

$K = \Pi_2\ D^o v^2\ \rho^1$

$$\Pi_2 = \frac{K}{\rho v^2}$$

$R = \Pi_1\, D^a\, v^b\, \rho^c$

$[R] = MLT^{-2}$
$[D] = L$
$[v] = LT^{-1}$
$[\rho] = ML^{-3}$

$MLT^{-2} = L^a (LT^{-1})^b (ML^{-3})^c$

$ML^1T^{-2} = L^{a+b-3c}M^c T^{-b}$

Time **-2 = -b** **b = 2**
Mass **c = 1**
Length 1 = a + b -3c
1 = a + 2 – 3 a = 2

$R = \Pi_1\ D^2 v^2\ \rho^1$

$$\Pi_1 = \frac{R}{\rho v^2 D^2}$$

It was shown earlier that the speed of sound in an elastic medium is given by the following formula.

$$a = (k/\rho)^{1/2}$$

It follows that $(k/\rho) = a^2$ and so $\Pi_2 = (a/v)^2$
The ratio v/a is called the Mach number (Ma) so $(Ma)^{-2}$
Π_1 is the Newton Number Ne.
The equation may be written as $\Pi_1 = \phi\Pi_2\, N_e$ or $Ne = \phi(M_a)$

SELF ASSESSMENT EXERCISE No. 2

1. The resistance to motion 'R' for a sphere of diameter 'D' moving at constant velocity 'v' on the surface of a liquid is due to the density 'ρ' and the surface waves produced by the acceleration of gravity 'g'. Show that the dimensionless equation linking these quantities is N_e = function(F_r)

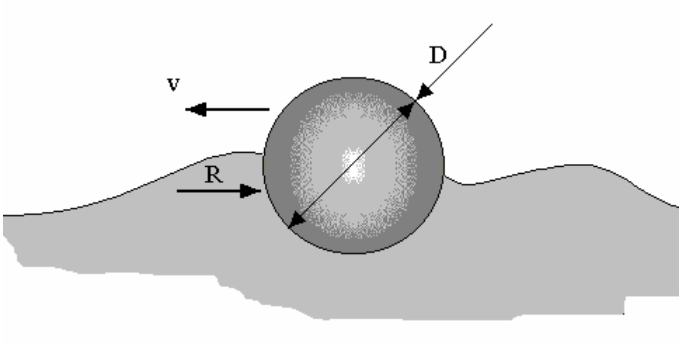


Figure 3

F_r is the Froude number and is given by $F_r = \sqrt{\frac{v^2}{gD}}$

Here is a useful tip. It is the power of g that cannot be found.

2. The Torque 'T' required to rotate a disc in a viscous fluid depends upon the diameter 'D' , the speed of rotation 'N' the density 'ρ' and the dynamic viscosity 'μ'. Show that the dimensionless equation linking these quantities is :

$$\{T\,D^{-5}\,N^{-2}\ \rho^{-1}\} = \text{function}\ \{\rho\,N\,D^{2}\ \ \mu^{-1}\}$$

EDEXCEL HIGHER

FLUID MECHANICS H1 UNIT 8

NQF LEVEL 4

OUTCOME 4 – HYDRAULIC MACHINES

TUTORIAL 6 – MOMENTUM AND PRESSURE FORCES

4 **Hydraulic machines**
***Impact of a jet*:** power of a jet; normal thrust on a moving flat vane; thrust on a moving hemispherical cup; velocity diagrams to determine thrust on moving curved vanes; fluid friction losses; system efficiency

***Operating principles*:** operating principles, applications and typical system efficiencies of common turbomachines including the Pelton wheel, Francis turbine and Kaplan turbine

***Operating principles of pumps*:** operating principles and applications of reciprocating and centrifugal pumps; head losses; pumping power; power transmitted; system efficiency

This is another major outcome requiring a lot of study time and the tutorial probably contains more than required.

On completion of this tutorial you should be able to solve the following.

- Forces due to pressure difference.
- Forces due to momentum changes.
- Forces on flat plates.
- Forces on curved vanes.
-

Let’s start with forces due to changes in the pressure of the fluid.

1. PRESSURE FORCES

Consider a duct as shown in fig.1. First identify the control volume on which to conduct a force balance. The inner passage is filled with fluid with pressure p1 at inlet and p2 at outlet. There will be forces on the outer surface of the volume due to atmospheric pressure. If the pressures of the fluid are measured relative to atmosphere (i.e. use gauge pressures) then these forces need not be calculated and the resultant force on the volume is due to that of the fluid only. The approach to be used here is to find the forces in both the x and y directions and then combine them to find the resultant force.

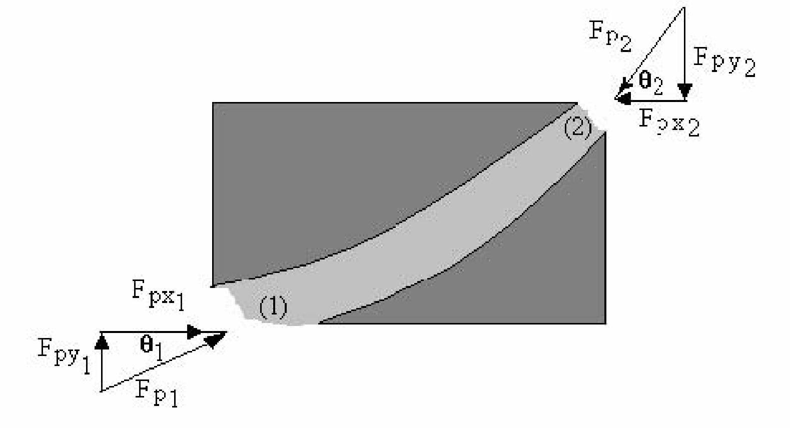


Fig.1

The force normal to the plane of the bore is pA.
At the inlet (1) the force is $Fp_1 = p_1A_1$
At the outlet (2) the force is $Fp_2 = p_2A_2$ These forces must be resolved vertically and horizontally to give the following. $Fpx1 = Fp1 \cos \theta 1$ (to the right) $Fpx2 = Fp1 \cos \theta 2$ (to the left) The total horizontal force $F_H = Fpx_1 - Fpx_2$
$Fpy_1 = Fp_1 \sin \theta_1$ (up) $Fpy_2 = Fp_2 \sin \theta_2$ (down) The total vertical force $F_V = Fpy_1 - Fpy_2$

WORKED EXAMPLE No. 1

A nozzle has an inlet area of 0.005 m^2 and it discharges into the atmosphere. The inlet gauge pressure is 3 bar. Calculate the resultant force on the nozzle.

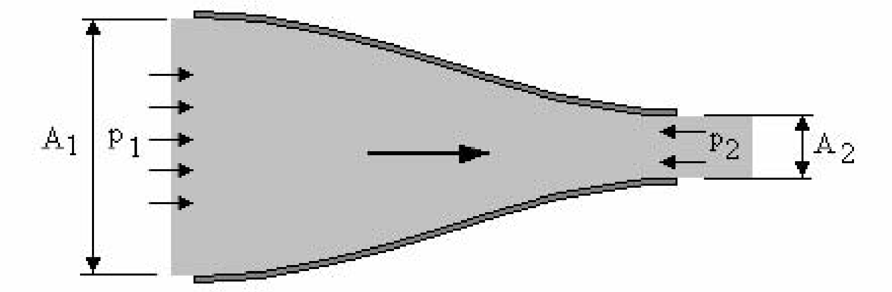


Fig.2

SOLUTION

Since the areas are only in the vertical plane, there is no vertical force. $F_V = 0$

Using gauge pressures, the pressure force at exit is zero. $F_{px2} = 0$
$F_{px1} = 3 \times 10^5 \times 0.005 = 1500$ N
$F_H = 1500 - 0 = 1500$ N to the right.

WORKED EXAMPLE No. 2

The nozzle shown has an inlet area of 0.002 m^2 and an outlet area of 0.0005 m^2. The inlet gauge pressure is 300 kPa and the outlet gauge pressure is 200 kPa. Calculate the horizontal and vertical forces on the nozzle.

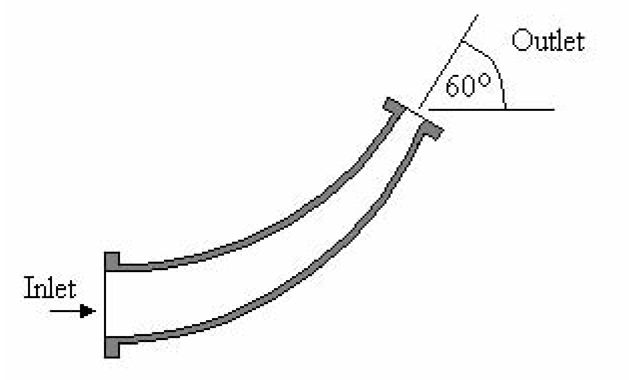


Fig.3

SOLUTION

$F_{p_1} = 300 \times 10^3 \times 0.002 = 600$ N

$F_{px_1} = 600$ N

$F_{py_1} = 0$ N since the plane is vertical.

$F_{p_2} = 200 \times 10^3 \times 0.0005 = 100$ N

$F_{px_2} = 100 \times \cos 60^o = 50$ N

$F_{py_2} = 100 \times \sin 60^o = 86.67$ N

Total Horizontal force $F_H = 600 - 50 = 550$ N

Total vertical force $F_V = 0 - 86.67$ N $= - 86.67$ N

2. MOMENTUM FORCES

When a fluid speeds up or slows down, inertial forces come into play. Such forces may be produced by either a change in the magnitude or the direction of the velocity since either change in this vector quantity produces acceleration.
For this section, we will ignore pressure forces and just study the forces due to velocity changes.

2.1 NEWTON'S 2nd LAW OF MOTION

This states that ***the change in momentum of a mass is equal to the impulse given to it.*** Impulse = Force x time Momentum = mass x velocity Change in momentum = Δmv
Newton's second law may be written as $\Delta mv = Ft$ Rearrange to make F the subject. $\Delta mv/t = F$
Since $\Delta v/t$ = acceleration 'a' we get the usual form of the law $F = ma$ The mass flow rate is m/t and at any given moment this is dm/dt or m' and for a constant flow rate,
only the velocity changes.
In fluids we usually express the second law in the following form. $F = (m/t)\, \Delta v = m'\Delta v$
$m'\Delta v$ is the rate of change of momentum so the second law may be restated as

F = Rate of change of momentum
F is the impulsive force resulting from the change. Δv is a vector quantity.

2.2 APPLICATION TO PIPE BENDS

Consider a pipe bend as before and use the idea of a control volume.

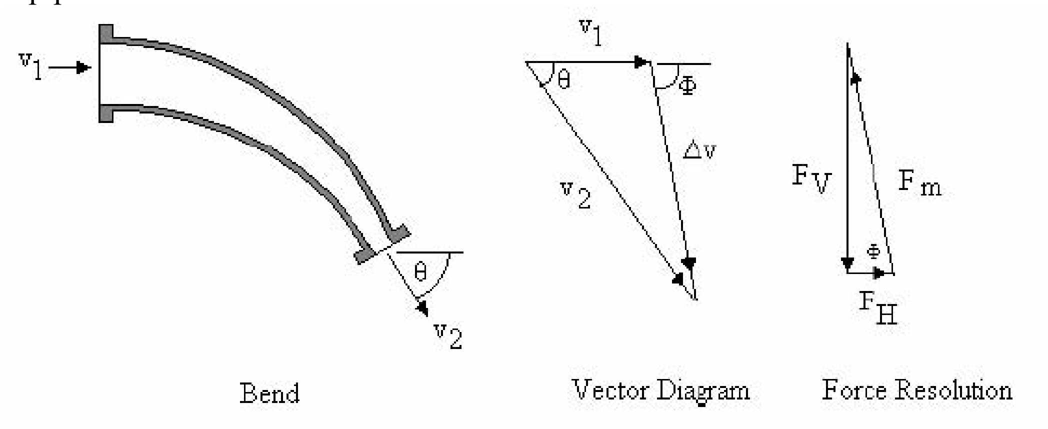


Fig.4

First find the vector change in velocity using trigonometry. $\mathbf{\tan\Phi = v2\sin\theta/(v2\cos\theta - v1)}$

$\mathbf{\Delta v = \{(v2\ \sin\theta)^2 + (v2\ \cos\theta - v1)^2\}^{1/2}}$ Alternatively Δv could be found by drawing the diagram to scale and measuring it. If we had no change in magnitude then v1 = v2 = v then

$$\mathbf{\Delta v = v\ \{2(1 - \cos\theta)\}^{1/2}}$$

The momentum force acting on the fluid is $F_m = m'\Delta v$ The force is a vector quantity which must be in the direction of Δv. Every force has an equal and opposite reaction so there must be a force on the bend equal and opposite to the force on the fluid. This force could be resolved vertically and horizontally such that $F_H = F_m\cos\Phi$ and $F_V = F_m\sin\Phi$

This theory may be applied to turbines and pump blade theory as well as to pipe bends.

SELF ASSESSMENT EXERCISE No. 1

1. A pipe bends through an angle of 90° in the vertical plane. At the inlet it has a cross sectional area of 0.003 m^2 and a gauge pressure of 500 kPa. At exit it has an area of 0.001 m^2 and a gauge pressure of 200 kPa.
   Calculate the vertical and horizontal forces due to the pressure only. (200 N and 1500 N).

2. A pipe bends through an angle of 45° in the vertical plane. At the inlet it has a cross sectional area of 0.002 m^2 and a gauge pressure of 800 kPa. At exit it has an area of 0.0008 m^2 and a gauge pressure of 300 kPa.
   Calculate the vertical and horizontal forces due to the pressure only. (169.7 N and 1430 N).

3. Calculate the momentum force acting on a bend of 130° which carries 2 kg/s of water at 16 m/s velocity.
   Determine the vertical and horizontal components. (Ans.24.5 N and 52.6 N)

4. Calculate the momentum force on a 180° bend that carries 5 kg/s of water. The pipe is 50 mm bore diameter throughout. The density is 1000 kg/m^3.
   (Ans. 25.46 N)

5. A horizontal pipe bend reduces from 300 mm bore diameter at inlet to 150 mm diameter at outlet. The bend is swept through 50o from its initial direction. The flow rate is 0.05 m3/s and the density is 1000 kg/m3. Calculate the momentum force on the bend and resolve it into two perpendicular directions relative to the initial direction. (Ans.108.1 N and 55.46 N).

3. COMBINED PRESSURE AND MOMENTUM FORCES

Now we will look at problems involving forces due to pressure changes and momentum changes at the same time. This is best done with a worked example since we have covered the theory already.

WORKED EXAMPLE No.3

A pipe bend has a cross sectional area of 0.01 m^2 at inlet and 0.0025 m^2 at outlet. It bends 90^o from its initial direction.
The velocity is 4 m/s at inlet with a pressure of 100 kPa gauge. The density is

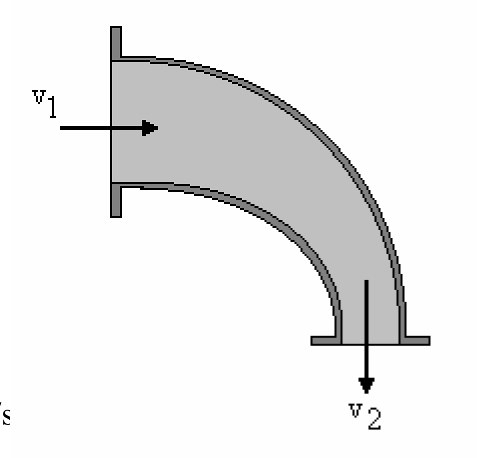


Fig.5

SOLUTION

$v_1 = 4$m/s Since $\rho A_1 v_1 = \rho A_2 v_2$ then $v_2 = 16$ m/s

We need the pressure at exit. This is done by applying Bernoulli between (1) and (2) as follows.

$p_1 + ½ \rho v_1^2 = p_2 + ½ \rho v_2^2$

$100 \times 10^3 + ½\ 1000 \times 4^2 = p_2 + 1000 \times ½\ 16^2$

$p_2 = 0$ kPa gauge

Now find the pressure forces.

$F_{px_1} = p_1 A_1 = 1200$ N

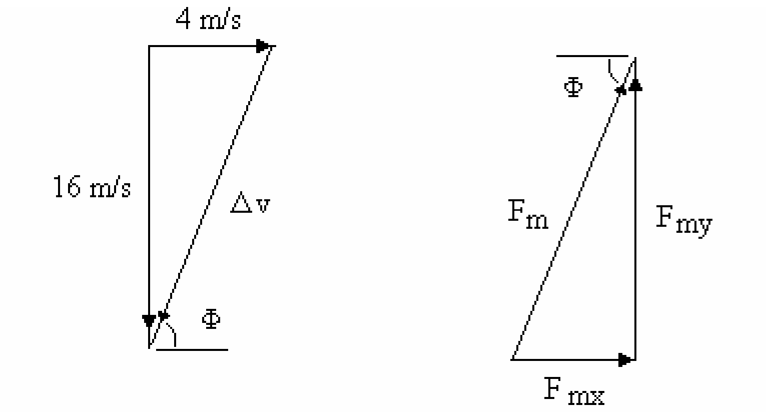
$F_{py_2} = p_2 A_2 = 0$ N Next solve the n

$m' = \rho A v = 40$ kg/s

$\Delta v = (4^2 + 16^2)^{½} = 16.49$ m/s

$F_m = m'\Delta v = 659.7$ N

$\phi = \tan^{-1}(16/4) = 75.96^o$



RESOLVE

Fig.6

$F_{my} = 659.7 \sin 75.96 = 640$ N $F_{mx} = 659.7 \cos 75.96 = 160$ N

Total forces in x direction = 1200 + 160 = 1360 N
Total forces in y direction = 0 + 640 = 640 N

ALTERNATIVE SOLUTION

Many people prefer to solve the complete problems by solving pressure and momentum forces in the x or y directions as follows.

x direction $m'v_1 + p_1 A_1 = F_X = 1200$ N

y direction $m'v_2 + p_2 A_2 = F_Y = 640$ N

When the bend is other than 90^o this has to be used more carefully because there is an x component at exit also.

4. **APPLICATIONS TO STATIONARY VANES**

When a jet of fluid strikes a stationary vane , the vane decelerates the fluid in a given direction. Even if the speed of the fluid is unchanged, a change in direction produces changes in the velocity vectors and hence momentum forces are produced. The resulting force on the vane being struck by the fluid is an ***impulsive force.*** Since the fluid is at atmospheric pressure at all times after leaving the nozzle, there are no forces due to pressure change.

4.1 FLAT PLATE NORMAL TO JET

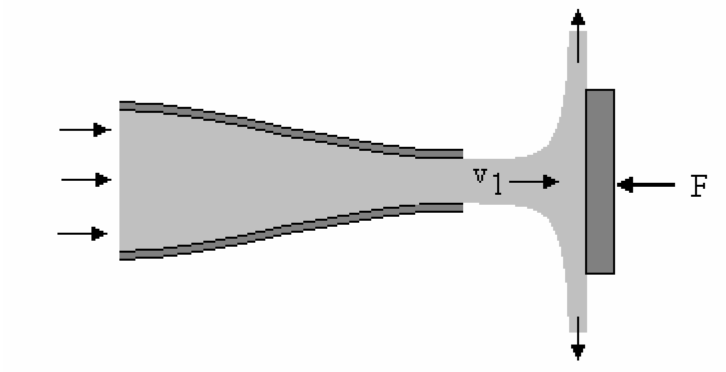


Fig.7

The velocity of the jet leaving the nozzle is v_1. The jet is decelerated to zero velocity in the original direction. Usually the liquid flows off sideways with equal velocity in all radial directions with no splashing occurring. The fluid is accelerated from zero in the radial directions but since the flow is equally divided no resultant force is produced in the radial directions. This means the only force on the plate is the one produced normal to the plate. This is found as follows.

m' = mass flow rate. Initial velocity = v_1.
Final velocity in the original direction = v_2 = 0.
Change in velocity $=\Delta v = v_2 - v_1 = -v_1$
Force = $m'\Delta v = -mv_1$

This is the force required to produce the momentum changes in the fluid. The force on the plate must be equal and opposite so

$$\mathbf{F = m'v_1 = \rho A\, v_1}$$

WORKED EXAMPLE No.4

A nozzle has an exit diameter of 15 mm and discharges water into the atmosphere. The gauge pressure behind the nozzle is 400 kPa. The coefficient of velocity is 0.98 and there is no contraction of the jet. The jet hits a stationary flat plate normal to its direction. Determine the force on the plate. The density of the water is 1000 kg/m^3. Assume the velocity of approach into the nozzle is negligible.

SOLUTION

The velocity of the jet is $v_1 = C_v(2\Delta p/\rho)^{1/2}$
$v_1 = 0.98\,(2 \times 400\,000/1000)^{1/2} = 27.72$ m/s
The nozzle exit area A = $\pi \times 0.015^2/4 = 176.7 \times 10^{-6}$ m^2.
The mass flow rate is $\rho A v_1 = 1000 \times 176.7 \times 10^{-6} \times 27.72 = 4.898$ kg/s.
The force on the vane = 4.898 x 27.72 = **135.8 N**

4.2 FLAT PLATE AT ANGLE TO JET

If the plate is at an angle as shown in fig. 4.9 then the fluid is not completely decelerated in the original direction but the radial flow is still equal in all radial directions. All the momentum normal to the plate is destroyed. It is easier to consider the momentum changes normal to the plate rather than normal to the jet.

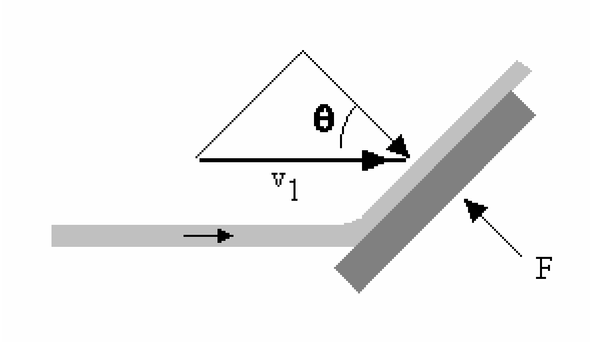


Fig. 8

Initial velocity normal to plate = $v_1 \cos\theta$.
Final velocity normal to plate = 0.
Force normal to plate = $m'\Delta v = 0 - \rho A v_1 \cos\theta$.
This is the force acting on the fluid so the force on the plate is

$$\mathbf{m' v_1 \cos\theta} \qquad \textbf{or} \qquad \boldsymbol{\rho A v_1^2 \cos\theta}.$$

If the horizontal and vertical components of this force are required then the force must be resolved.

WORKED EXAMPLE No. 5

A jet of water has a velocity of 20 m/s and flows at 2 kg/s. The jet strikes a stationary flat plate. The normal direction to the plate is inclined at 30^o to the jet. Determine the force on the plate in the direction of the jet.

SOLUTION

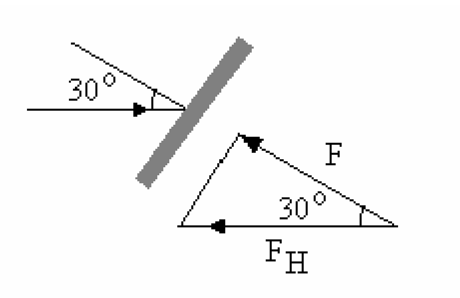


Fig. 9

The force normal to the plate is $mv_1 \cos\theta = 2 \times 20\cos 30^o = 34.64$ N.

The force in the direction of the jet is found by resolving.

$F_H = F/\cos30^o = 34.64/\cos 30^o = 40$ N

4.3 CURVED VANES

When a jet hits a curved vane, it is usual to arrange for it to arrive on the vane at the same angle as the vane. The jet is then diverted from with no splashing by the curve of the vane. If there is no friction present, then only the direction of the jet is changed, not its speed.

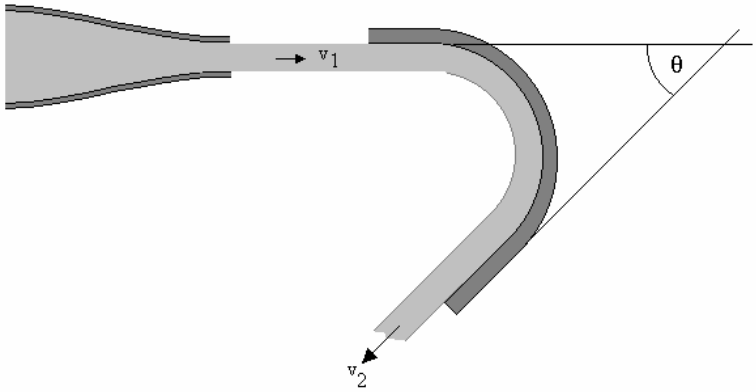
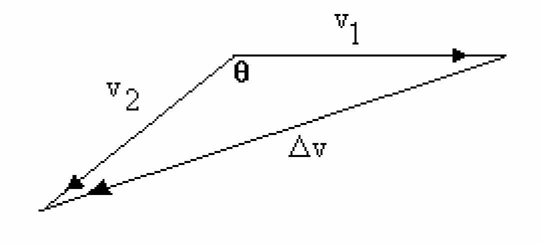


Fig. 10

This is the same problem as a pipe bend with uniform size. v_1 is numerically equal to v_2.



If the deflection angle is θ as shown in figs. 10 and 11 then the impulsive force is

$$\mathbf{F = m'\Delta v = m' v_1\{2(1 - \cos\theta)\}^{1/2}}$$

Fig. 11

The direction of the force on the fluid is in the direction of Δv and the direction of the force on the vane is opposite. The force may be resolved to find the forces acting horizontally and/or vertically.

It is often necessary to solve the horizontal force and this is done as follows.

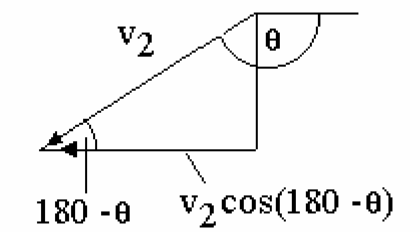


Fig.12

Initial horizontal velocity = $v_{H1} = v_1$
Final horizontal velocity = $v_{H2} = -v_2 \cos (180 - \theta) = v_2 \cos \theta$
Change in horizontal velocity = Δv_{H1}
Since $v_2 = v_1$ this becomes $\Delta v_h = \{v_2 \cos\theta - v_1\} = v_1\{\cos\theta - 1\}$
Horizontal force on fluid = $m'v_1\{\cos\theta - 1\}$
The horizontal force on the vane is opposite so

Horizontal force = $m'\Delta v_H = m'v_1\{1 - \cos\theta\}$

WORKED EXAMPLE No. 6

A jet of water travels horizontally at 16 m/s with a flow rate of 2 kg/s. It is deflected 130^o by a curved vane. Calculate resulting force on the vane in the horizontal direction.

SOLUTION

The resulting force on the vane is $F = m' v_1\{2(1 - \cos\theta)^{½}$

$F = 2 \times 16 \{2(1 - \cos 130^o)\}^{½} = 58$ N

The horizontal force is

$F_H = m' v_1\{\cos\theta - 1\}$
$F_H = 2 \times 16 \times (1 - \cos 130)$
$F_H = 52.6$ N

SELF ASSESSMENT EXERCISE No. 2

Assume the density of water is 1000 kg/m^3 throughout.

1. A pipe bends through 90^o from its initial direction as shown in fig. 4.7. The pipe reduces in diameter such that the velocity at point (2) is 1.5 times the velocity at point (1). The pipe is 200 mm diameter at point (1) and the static pressure is 100 kPa. The volume flow rate is 0.2 m^3/s. Assuming that there is no friction calculate the following.

   a) The static pressure at (2).

   b) The velocity at (2).

   c) The horizontal and vertical forces on the bend F_H and F_V.

   d) The total resultant force on the bend.

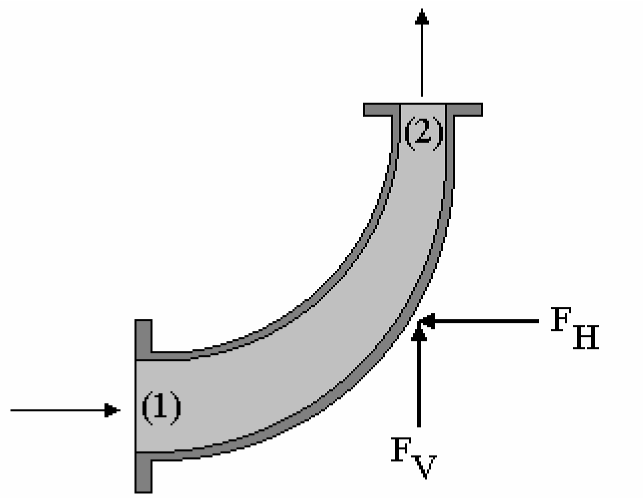


Fig. 13

2. A nozzle produces a jet of water. The gauge pressure behind the nozzle is 2 MPa. The exit diameter is 100 mm. The coefficient of velocity is 0.97 and there is no contraction of the jet. The approach velocity is negligible. The jet of water is deflected 165^o from its initial direction by a stationary vane. Calculate the resultant force on the nozzle and on the vane due to momentum changes only. (29.5 kN and 58.5 kN).

3. A stationary vane deflects 5 kg/s of water 50^o from its initial direction. The jet velocity is 13 m/s. Draw the vector diagram to scale showing the velocity change. Deduce by either scaling or calculation the change in velocity and go on to calculate the force on the vane in the original direction of the jet. (49.8 N).

4. A jet of water travelling with a velocity of 25 m/s and flow rate 0.4 kg/s is deflected 150^o from its initial direction by a stationary vane. Calculate the force on the vane acting parallel to and perpendicular to the initial direction. (Ans.18.66 N and 5 N)

5. A jet of water discharges from a nozzle 30 mm diameter with a flow rate of 15 dm^3/s into the atmosphere. The inlet to the nozzle is 100 mm diameter. There is no friction nor contraction of the jet. Calculate the following.

a) The jet velocity.
b) The gauge pressure at inlet.
c) The force on the nozzle.

The jet strikes a flat stationary plate normal to it. Determine the force on the plate.

5. MOVING VANES

When a vane moves away from the jet as shown on fig.4.14, the mass flow arriving on the vane is reduced because some of the mass leaving the nozzle is producing a growing column of fluid between the jet and the nozzle. This is what happens in turbines where the vanes are part of a revolving wheel. We need only consider the simplest case of movement in a straight line in the direction of the jet.

5.1 MOVING FLAT PLATE

The velocity of the jet is v and the velocity of the vane is u. If you were on the plate, the velocity of the fluid arriving would be v - u. This is the relative velocity, that is, relative to the plate. The mass flow rate arriving on the plate is then

$$\mathbf{m' = \rho A(v-u)}$$

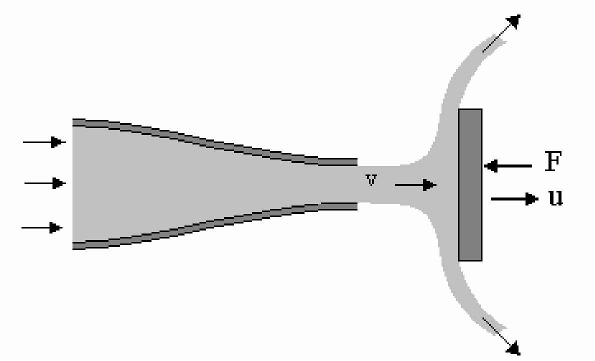


Fig. 14

The initial direction of the fluid is the direction of the jet. However, due to movement of the plate, the velocity of the fluid as it leaves the edge is not at 90^o to the initial direction. In order to understand this we must consider the fluid as it flows off the plate. Just before it leaves the plate it is still travelling forward with the plate at velocity u. When it leaves the plate it will have a true velocity that is a combination of its radial velocity and u. The result is that it appears to come off the plate at a forward angle as shown.

We are only likely to be interested in the force in the direction of movement so we only require the change in velocity of the fluid in this direction.

The initial forward velocity of the fluid = v
The final forward velocity of the fluid = u
The change in forward velocity = v-u
The force on the plate = m'ρv = m' (v-u)
Since m' = ρA(v-u) then the force on the plate is

$$\mathbf{F = \rho A(v-u)^2}$$

5.2 MOVING CURVED VANE

Turbine vanes are normally curved and the fluid joins it at the same angle as the vane as shown in the diagram.

The velocity of the fluid leaving the nozzle is v_1. This is a true or absolute velocity as observed by anyone standing still on the ground.

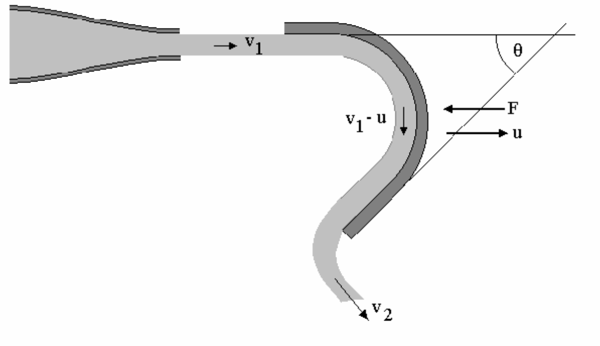


Fig.15

The fluid arrives on the vane with relative velocity v_1-u as before. This is a relative velocity as observed by someone moving with the vane. If there is no friction then the velocity of the fluid over the surface of the vane will be v_1-u at all points. At the tip where the fluid leaves the vane, it will have two velocities. The fluid will be flowing at v_1-u over the vane but also at velocity u in the forward direction. The true velocity v_2 at exit must be the vector sum of these two.

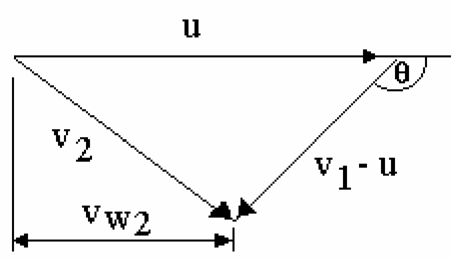


Fig. 16

If we only require the force acting on the vane in the direction of movement then we must find the horizontal component of v_2. Because this direction is the direction in which the vane is whirling about the centre of the wheel, it is called the velocity of whirl v_{w2}. The velocity v_1 is also in the direction of whirling so it follows that $v_1 = v_{w1}$.

V_{w2} may be found by drawing the vector diagram (fig.4.16) to scale or by using trigonometry. In this case you may care to show for yourself that $v_{w2} = u + (v_1-u)(\cos\theta)$

The horizontal force on the vane becomes $F_H = m' (v_{w1}-v_{w2}) = m' (v_1-v_{w2})$

You may care to show for yourself that this simplifies down to $Fh = m'(v_1-u)(1-\cos\theta)$

This force moves at the same velocity as the vane. The power developed by a force is the product of force and velocity. This is called the Diagram Power (D.P.) and the diagram power developed by a simple turbine blade is

$$\text{D.P.} = m'u(v_1-u)(1-\cos\theta)$$

This work involving the force on a moving vane is the basis of turbine problems and the geometry of the case considered is that of a simple water turbine known as a Pelton Wheel. You are not required to do this in the exam. It is unlikely that the examination will require you to calculate the force on the moving plate but the question in self assessment exercise 5 does require you to calculate the exit velocity v_2.

WORKED EXAMPLE No. 7

A simple turbine vane as shown in fig.15 moves at 40 m/s and has a deflection angle of 150^o. The jet velocity from the nozzle is 70 m/s and flows at 1.7 kg/s.

Calculate the absolute velocity of the water leaving the vane and the diagram power.

SOLUTION

Drawing the vector diagram (fig.4.15) to scale, you may show that v_2 = 20.5 m/s. This may also be deduced by trigonometry. The angle at which the water leaves the vane may be measured from the diagram or deduced by trigonometry and is 46.9^o to the original jet direction.

D.P. = $m'u(v_1-u)(1+\cos\theta)$ = 1.7 x 40(70-40)(1 - cos 150) = 3807 Watts

SELF ASSESSMENT EXERCISE No. 3

1. A vane moving at 30 m/s has a deflection angle of 90^o. The water jet moves at 50 m/s with a flow of 2.5 kg/s. Calculate the diagram power assuming that all the mass strikes the vane.
(1.5 kW).

2. Figure 10 shows a jet of water 40 mm diameter flowing at 45 m/s onto a curved fixed vane. The deflection angle is 150^o. There is no friction. Determine the magnitude and direction of the resultant force on the vane.

   The vane is allowed to move away from the nozzle in the same direction as the jet at a velocity of 18 m/s. Draw the vector diagram for the velocity at exit from the vane and determine the magnitude and direction of the velocity at exit from the vane.

EDEXCEL HIGHER

FLUID MECHANICS H1 UNIT 8

NQF LEVEL 4

OUTCOME 4 – HYDRAULIC MACHINES

TUTORIAL 7 – TURBINES AND PUMPS

4 **Hydraulic machines**
Impact of a jet**:** power of a jet; normal thrust on a moving flat vane; thrust on a moving hemispherical cup; velocity diagrams to determine thrust on moving curved vanes; fluid friction losses; system efficiency

Operating principles**:** operating principles, applications and typical system efficiencies of common turbomachines including the Pelton wheel, Francis turbine and Kaplan turbine

Operating principles of pumps**:** operating principles and applications of reciprocating and centrifugal pumps; head losses; pumping power; power transmitted; system efficiency

This is another major outcome requiring a lot of study time and the tutorial probably contains more than required.

On completion of this tutorial you should be able to solve the following.

- Explain the general principles of pumps and turbines.
- Construct and analyse vector diagrams for pump and turbine rotors.
- Explain and analyses the Pelton Wheel.
- Explain the Francis Turbine.
- Explain and analyses the Kaplan turbine
- Explain and analyses the Centrifugal Pump.

Let's start with forces due to changes in the pressure of the fluid.

1. TURBINES

A water turbine is a device for converting water (fluid) power into shaft (mechanical) power. A pump is a device for converting shaft power into water power.

Two basic categories of machines are the rotary type and the reciprocating type. Reciprocating motors are quite common in power hydraulics but the rotary principle is universally used for large power devices such as on hydroelectric systems.

Large pumps are usually of the rotary type but reciprocating pumps are used for smaller applications.

1.1 GENERAL PRINCIPLES OF TURBINES.

WATER POWER

This is the fluid power supplied to the machine in the form of pressure and volume.
Expressed in terms of pressure head the formula is W.P. = $mg\Delta H$
M is the mass flow rate in kg/s and ΔH is the pressure head difference over the turbine in metres.
Remember that $\Delta p = \rho g \Delta H$
Expressed in terms of pressure the formula is W.P. = $Q\Delta p$
Q is the volume flow rate in m^3/s. Δp is the pressure drop over the turbine in N/m^2 or Pascals.

SHAFT POWER

This is the mechanical, power output of the turbine shaft. The well known formula is
S.P. = $2\pi NT$ Where T is the torque in Nm and N is the speed of rotation in rev/s

DIAGRAM POWER

This is the power produced by the force of the water acting on the rotor. It is reduced by losses before appearing as shaft power. The formula for D.P. depends upon the design of the turbine and involves analysis of the velocity vector diagrams.

HYDRAULIC EFFICIENCY η_{hyd}

This is the efficiency with which water power is converted into diagram power and is given by

$$\eta_{hyd} = \text{D.P./W.P.}$$

MECHANICAL EFFICIENCY η_{mech}

This is the efficiency with which the diagram power is converted into shaft power. The difference is the mechanical power loss.

$$\eta_{mech} = \text{S.P./D.P.}$$

OVERALL EFFICIENCY $\eta_{o/a}$

This is the efficiency relating fluid power input to shaft power output.

$$\eta_{o/a} = \text{S.P./W.P.}$$

It is worth noting at this point that when we come to examine pumps, all the above expressions are inverted because the energy flow is reversed in direction.
The water power is converted into shaft power by the force produced when the vanes deflect the direction of the water. There are two basic principles in the process, ***IMPULSE and REACTION.***

IMPULSE occurs when the direction of the fluid is changed with no pressure change. It follows that the magnitude of the velocity remains unchanged.

REACTION occurs when the water is accelerated or decelerated over the vanes. A force is needed to do this and the reaction to this force acts on the vanes.
Impulsive and reaction forces are determined by examining the changes in velocity (magnitude and direction) when the water flows over the vane. The following is a typical analysis.
The vane is part of a rotor and rotates about some centre point. Depending on the geometrical layout, the inlet and outlet may or may not be moving at the same velocity and on the same circle. In order to do a general study, consider the case where the inlet and outlet rotate on two different diameters and hence have different velocities.

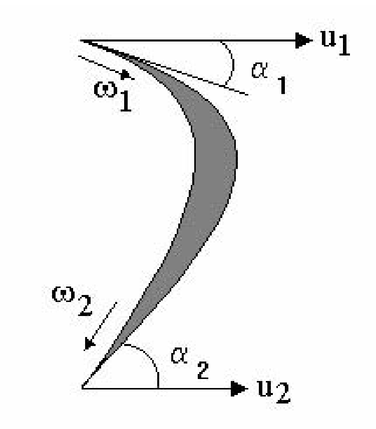


Fig. 1

u_1 is the velocity of the blade at inlet and u_2 is the velocity of the blade at outlet. Both have tangential directions.ω_1 is the relative velocity at inlet and ω_2 is the relative velocity at outlet.

The water on the blade has two velocity components. It is moving tangentially at velocity u and over the surface at velocity ω. The absolute velocity of the water is the vector sum of these two and is denoted v. At any point on the vane $v = \omega + u$

At inlet, this rule does not apply unless the direction of v_1 is made such that the vector addition is true. At any other angle, the velocities will not add up and the result is chaos with energy being lost as the water finds its way onto the vane surface. The perfect entry is called ***"SHOCKLESS ENTRY"*** and the entry angle β_1 must be correct. This angle is only correct for a given value of v_1.

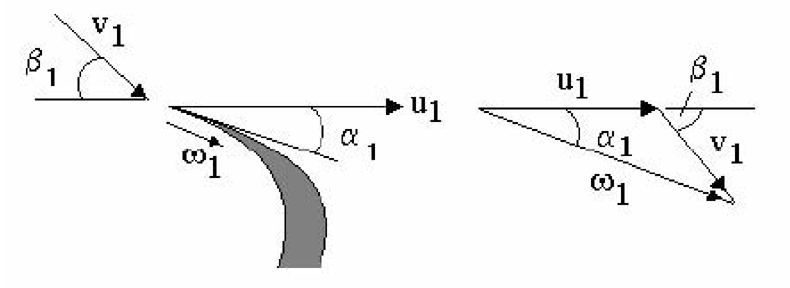


Fig.2

<u>**INLET DIAGRAM**</u>

For a given or fixed value of u_1 and v_1, shockless entry will occur only if the vane angle α_1 is correct or the delivery angle β_1 is correct. In order to solve momentum forces on the vane and deduce the flow rates, we are interested in two components of v_1. These are the components in the direction of the vane movement denoted v_w (meaning velocity of whirl) and the direction at right angles to it v_R (meaning radial velocity but it is not always radial in direction depending on the wheel design). The suffix (1) indicates the entry point. A typical vector triangle is shown.

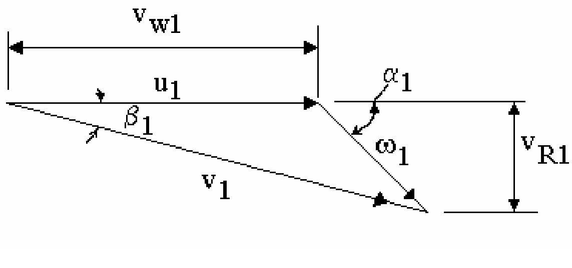


Fig.3

OUTLET DIAGRAM

At outlet, the absolute velocity of the water has to be the vector resultant of u and ω and the direction is unconstrained so it must come off the wheel at the angle resulting. Suffix (2) refers to the outlet point. A typical vector triangle is shown.

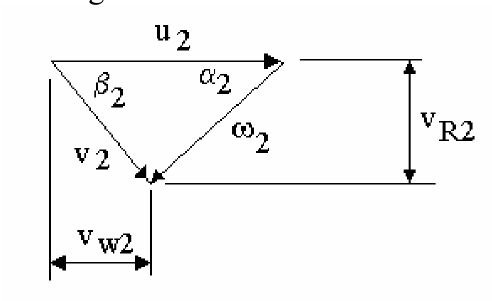


Fig. 4

DIAGRAM POWER

Diagram power is the theoretical power of the wheel based on momentum changes in the fluid. The force on the vane due to the change in velocity of the fluid is $F = m\Delta v$ and these forces are vector quantities. m is the mass flow rate. The force that propels the wheel is the force developed in the direction of movement (whirl direction). In order to deduce this force, we should only consider the velocity changes in the whirl direction (direction of rotation) Δv_w. The power of the force is always the product of force and velocity. The velocity of the force is the velocity of the vane (u). If this velocity is different at inlet and outlet it can be shown that the resulting power is given by

$$D.P. = m\,\Delta v_w = m\,(u_1 v_{w1} - u_2\, v_{w2})$$

1.2 PELTON WHEEL

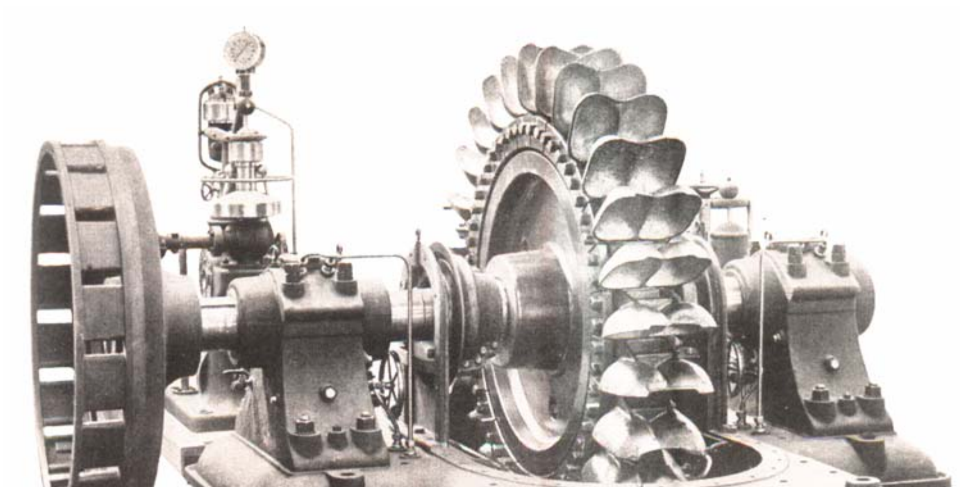


Fig. 5 Pelton wheel with the casing removed

Pelton wheels are mainly used with high pressure heads such as in mountain hydroelectric schemes. The diagram shows a layout for a Pelton wheel with two nozzles.

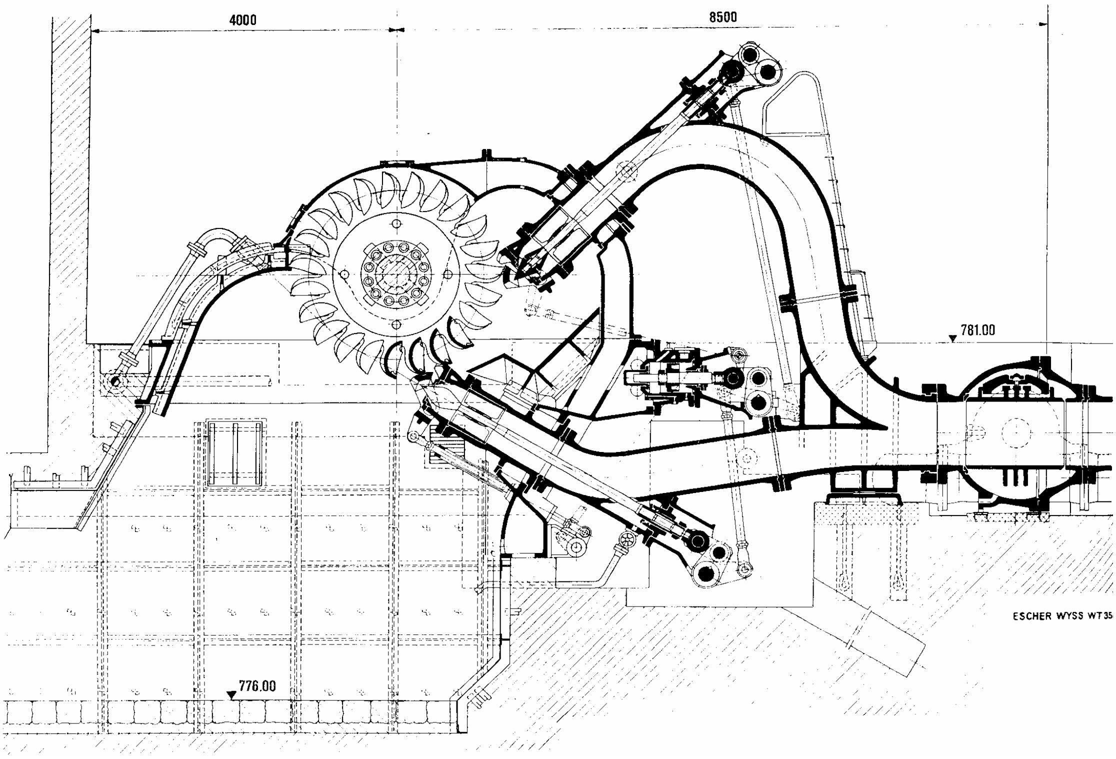


Fig. 6 Typical Layout

The Pelton Wheel is an impulse turbine. The fluid power is converted into kinetic energy in the nozzles. The total pressure drop occurs in the nozzle. The resulting jet of water is directed tangentially at buckets on the wheel producing impulsive force on them. The buckets are small compared to the wheel and so they have a single velocity $u = \pi ND$ D is the mean diameter of rotation for the buckets.

The theoretical velocity issuing from the nozzle is given by $v_1 = (2gH)^{1/2}$ or $v_1 = (2p/\rho)^{1/2}$

Allowing for friction in the nozzle this becomes $v_1 = C_v(2gH)^{1/2}$ or $v_1 = C_v(2p/\rho)^{1/2}$
H is the gauge pressure head behind the nozzle, p the gauge pressure and c_v the coefficient of velocity and this is usually close to unity.

The mass flow rate from the nozzle is $m = C_c\,\rho A v_1 = C_c \rho A C_v (2gH)^{1/2} = C_d\,\rho A(2gH)^{1/2}$

C_c is the coefficient of contraction (normally unity because the nozzles are designed not to have a contraction).

C_d is the coefficient of discharge and
$C_d = C_c\, C_v$

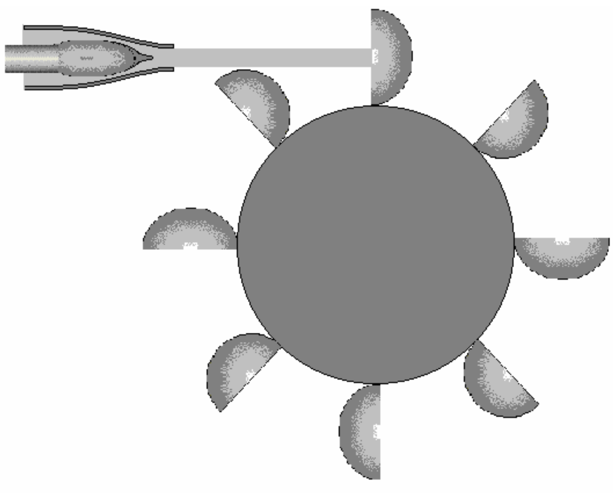
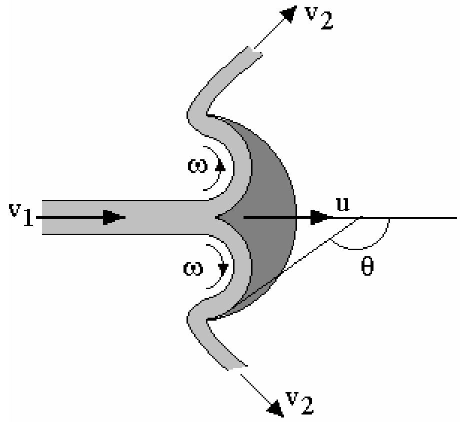


Fig.7 Layout of Pelton wheel with one nozzle

In order to produce no axial force on the wheel, the flow is divided equally by the shape of the bucket. This produces a zero net change in momentum in the axial direction. The water is deflected over each half of the bucket by an angle of θ degrees. Since the change in momentum is the same for both halves of the flow, we need only consider the vector diagram for one half. The initial velocity is v1 and the bucket velocity u1 is in the same direction. The relative velocity of the water at inlet (in the middle) is ω_1 and is also in the same direction so the vector diagram is a straight line.



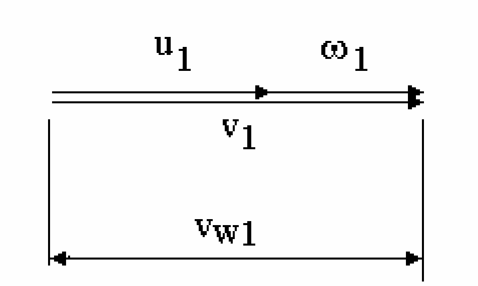


Fig. 9 Vector Diagram

Fig.8 Cross section through bucket

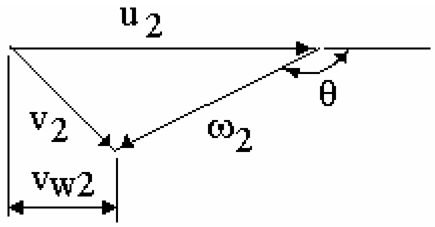
If the water is not slowed down as it passes over the bucket surface, the relative velocity ω_2 will be the same as ω_1. In reality friction slows it down slightly and we define a blade friction coefficient as

$$k = \omega_2/\omega_1$$

The exact angle at which the water leaves the sides of the bucket depends upon the other velocities but as always the vectors must add up so that $v_2 = u + \omega_2$

Note that $u_2 = u_1 = u$ since the bucket has a uniform velocity everywhere.

It is normal to use ω_1 and u as common to both diagrams and combine them as shown.



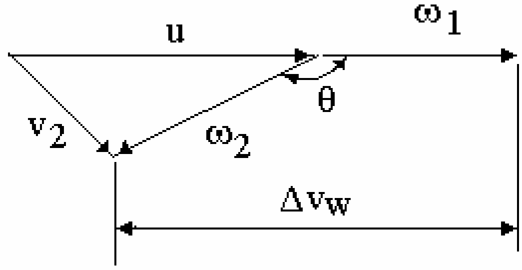


Fig. 10 Inlet Vector Diagram

Fig. 11 Combined Vector Diagarm

Since $u_2 = u_1 = u$ the diagram power becomes D.P. = $mu\Delta v_W$

Examining the combined vector diagram shows that $\Delta v_W = \omega_1 - \omega_2\cos\theta$

Hence D.P.= $mu(\omega_1 - \omega_2\cos\theta)$ but $\omega_2 = k\omega_1$

D.P.= $mu\omega_1(1 - k\cos\theta)$ but $\omega_1 = v_1 - u$

D.P.=$mu(v_1 - u)(1 - k\cos\theta)$

WORKED EXAMPLE No. 1

A Pelton wheel is supplied with 1.2 kg/s of water at 20 m/s. The buckets rotate on a mean diameter of 250 mm at 800 rev/min. The deflection angle is 165^o and friction is negligible. Determine the diagram power. Draw the vector diagram to scale and determine □v_W.

SOLUTION

$u = \pi ND/60 = \pi \times 800 \times 0.25/60 = 10.47$ m/s

D.P = $mu(v_1-u)(1- k\cos\theta)$

D.P = 1.2 x 10.47 x (20 - 10.47)(1 - cos 165) = 235 Watts

You should now draw the vector diagram to scale and show that Δv_W= 18.5 m/s

SELF ASSESSMENT EXERCISE No. 1

1. The buckets of a Pelton wheel revolve on a mean diameter of 1.5 m at 1500 rev/min. The jet velocity is 1.8 times the bucket velocity. Calculate the water flow rate required to produce a power output of 2MW. The mechanical efficiency is 80% and the blade friction coefficient is 0.97. The deflection angle is 165^o.
(Ans. 116.3 kg/s)

2. Calculate the diagram power for a Pelton Wheel 2m mean diameter revolving at 3000 rev/min with a deflection angle of 170^o under the action of two nozzles , each supplying 10 kg/s of water with a velocity twice the bucket velocity. The blade friction coefficient is 0.98. (Ans. 3.88 MW)

   If the coefficient of velocity is 0.97, calculate the pressure behind the nozzles. (Ans 209.8 MPa)

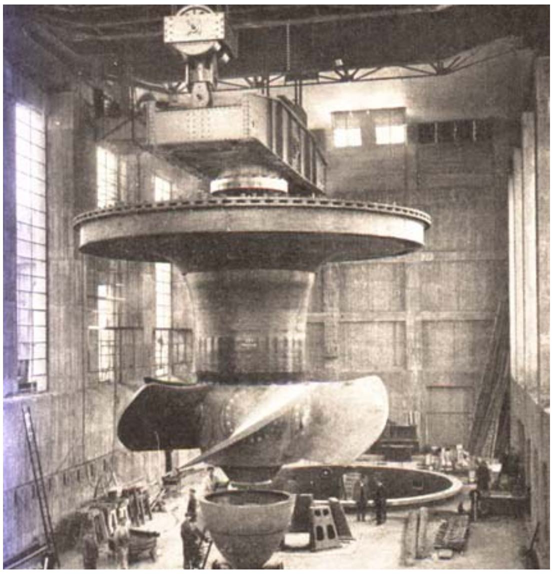
3. A Pelton Wheel is 1.7 m mean diameter and runs at maximum power. It is supplied from two nozzles. The gauge pressure head behind each nozzle is 180 metres of water. Other data for the wheel is :

   Coefficient of Discharge C_d= 0.99
   Coefficient of velocity C_v= 0.995
   Deflection angle = 165^o.
   Blade friction coefficient = 0.98
   Mechanical efficiency = 87%
   Nozzle diameters = 30 mm

   Calculate the following.

a) The jet velocity (59.13 m/s)
b) The mass flow rate (41.586 kg/s)
c) The water power (73.432 kW)
d) The diagram power (70.759 kW)
e) The diagram efficiency (96.36%)
f) The overall efficiency (83.8%)
g) The wheel speed in rev/min (332 rev/min)

1.4 **KAPLAN TURBINE**



The Kaplan turbine is a pure reaction turbine. The main point concerning this is that all the flow energy and pressure is expended over the rotor and not in the supply nozzles. The picture shows the rotor of a large Kaplan turbine. They are most suited to low pressure heads and large flow rates such as on dams and tidal barrage schemes.

The diagram below shows the layout of a large hydroelectric generator in a dam.

Fig. 12 Picture of a Kaplan Turbine Rotor

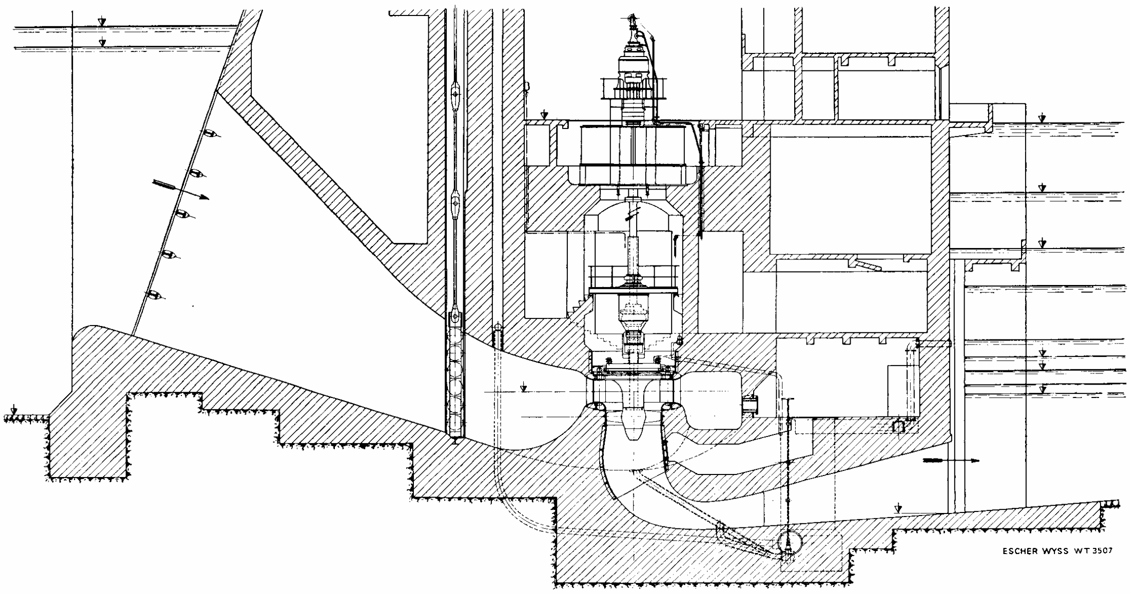


Fig. 13 Typical Layout

1.5 **FRANCIS WHEEL**

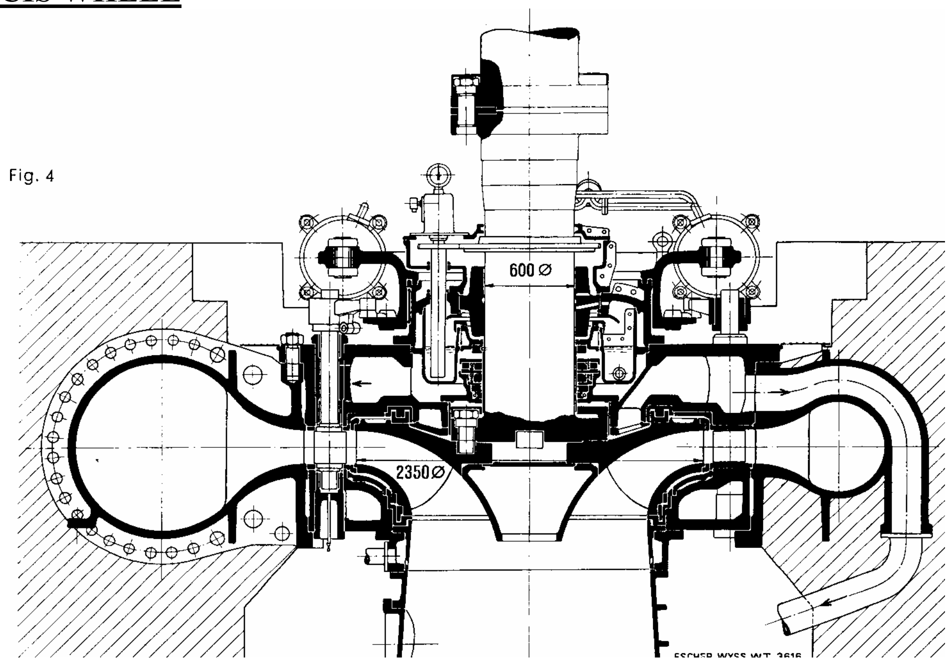
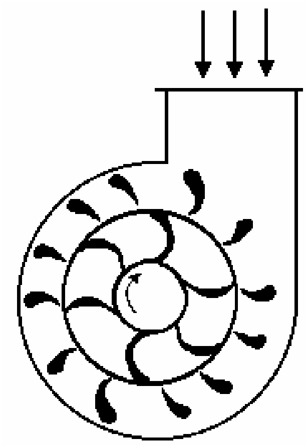


Fig. 14



The Francis wheel is an example of a mixed impulse and reaction turbine. They are adaptable to varying heads and flows and may be run in reverse as a pump such as on a pumped storage scheme. The diagram shows the layout of a vertical axis Francis wheel.

The Francis Wheel is an inward flow device with the water entering around the periphery and moving to the centre before exhausting. The rotor is contained in a casing that spreads the flow and pressure evenly around the periphery.

Fig. 15

The impulse part comes about because guide vanes are used to produce an initial velocity v_1 that is directed at the rotor. Pressure drop occurs in the guide vanes and the velocity is $v_1 = k\,(\Delta H)^{½}$ where ΔH is the head drop in the guide vanes.

The angle of the guide vanes is adjustable so that the inlet angle β_1 is correct for shockless entry.

The shape of the rotor is such that the vanes are taller at the centre than at the ends. This gives control over the radial velocity component and usually this is constant from inlet to outlet. The volume flow rate is usually expressed in terms of radial velocity and circumferential area.

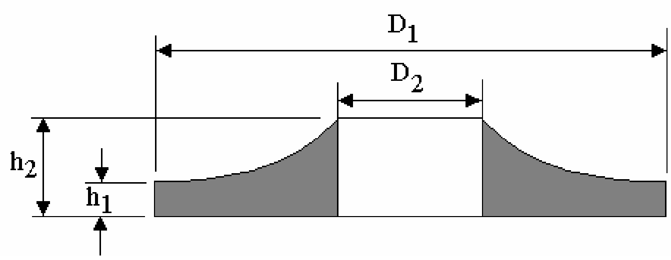


Fig. 16

v_R = radial velocity A = circumferential area = D h k
$Q = v_R\, \pi D\, h\, k$ h = height of the vane.

k is a factor which allows for the area taken up by the thickness of the vanes on the circumference. If v_R is constant then since Q is the same at all circumferences, $D_1h_1=D_2h_2$.

VECTOR DIAGRAMS

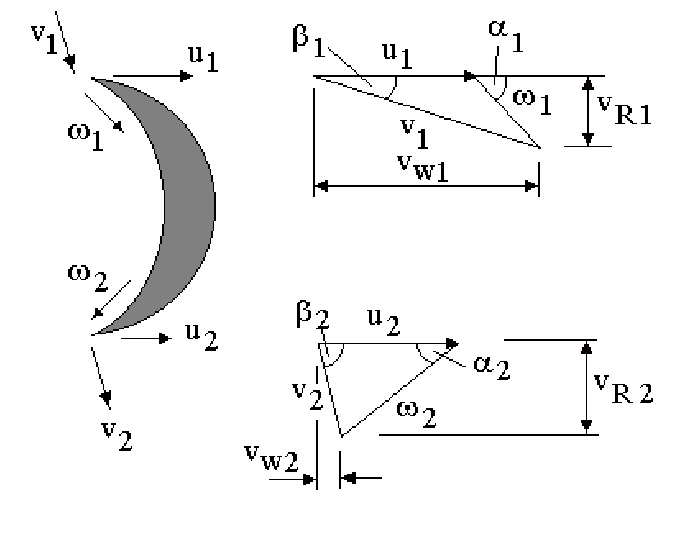


Fig. 17

The diagram shows how the vector diagrams are constructed for the inlet and outlet. Remember the rule is that the vectors add up so that $u + v = \omega$

If u is drawn horizontal as shown, then V_W is the horizontal component of v and v_R is the radial component (vertical).

MORE DETAILED EXAMINATION OF VECTOR DIAGRAM

Applying the sine rule to the inlet triangle we find

$$\frac{v_1}{\sin(180-\alpha_1)} = \frac{u_1}{\sin\{180-\beta_1-(180-\alpha_1)\}}$$

$$\frac{v_1}{\sin(\alpha_1)} = \frac{u_1}{\sin(\alpha_1-\beta_1)} \qquad v_1 = \frac{u_1\sin(\alpha_1)}{\sin(\alpha_1-\beta_1)}.................(1)$$

$$v_1 = \frac{v_{r1}}{\sin(\beta_1)}...........(2) \qquad v_{r1} = v_{w1}\tan\beta_1................(3) \qquad \text{equate (1) and (2)}$$

$$\frac{u_1\sin(\alpha_1)}{\sin(\alpha_1-\beta_1)} = \frac{v_{r1}}{\sin(\beta_1)} \qquad v_{r1} = \frac{u_1\sin(\alpha_1)\sin(\beta_1)}{\sin(\alpha_1-\beta_1)}...........(4) \qquad \text{equate (3) and (4)}$$

$$v_{w1} = \frac{u_1\sin(\alpha_1)\sin(\beta_1)}{\sin(\alpha_1-\beta_1)\tan\beta_1}.............(5)$$

If all the angles are known, then v_{w1} may be found as a fraction of u_1.

DIAGRAM POWER

Because u is different at inlet and outlet we express the diagram power as :

$$D.P. = m\,\Delta(uv_w) = m\,(u_1v_{w1} - u_2v_{w2})$$

The kinetic energy represented by v_2 is energy lost in the exhausted water. For maximum efficiency, this should be reduced to a minimum and this occurs when the water leaves radially with no whirl so that $v_{w2} = 0$. This is produced by designing the exit angle to suit the speed of the wheel. The water would leave down the centre hole with some swirl in it. The direction of the swirl depends upon the direction of v_2 but if the flow leaves radially, there is no swirl and less kinetic energy. Ideally then,

$$D.P. = m\,u_1\,v_{w1}$$

WATER POWER

The water power supplied to the wheel is $mg\Delta H$ where ΔH is the head difference between inlet and outlet.

HYDRAULIC EFFICIENCY

The maximum value with no swirl at exit is $\eta_{hyd} = D.P./W.P. = u_1 v_{w1}/g\rho H$

OVERALL EFFICIENCY

$$\eta_{o/a} = \text{Shaft Power/Water Power}$$
$$\eta_{o/a} = 2\pi NT/mg\Delta H$$

LOSSES

The hydraulic losses are the difference between the water power and diagram power.

Loss = $mg\Delta H - mu_1 v_{w1} = mgh_L$

$h_L = \Delta H - u_1 v_{w1}/g$

$\Delta H - h_L = u_1 v_{w1}/g$

SELF ASSESSMENT EXERCISE No. 2

You have studied the basic principles of Pelton , Kaplan and Francis turbines.

Hydroelectric schemes may have very high pressures (e.g. mountain lakes). They may have very low pressures (e.g. dammed lakes). The pressure head may vary (e.g. tidal barrage schemes). They may have access to large or small quantities of water. Sometimes they are used as pumps (e.g. pumped storage schemes).

Find out what each turbine is best suited to. Explain what it is in their design that suit to them to their application.

2. CENTRIFUGAL PUMPS

2.1 GENERAL THEORY

A Centrifugal pump is a Francis turbine running backwards. The water between the rotor vanes experiences centrifugal force and flows radially outwards from the middle to the outside. As it flows, it gains kinetic energy and when thrown off the outer edge of the rotor, the kinetic energy must be converted into flow energy. The use of vanes similar to those in the Francis wheel helps. The correct design of the casing is also vital to ensure efficient low friction conversion from velocity to pressure. The water enters the middle of the rotor without swirling so we know v_{w1} is always zero for a c.f. pump. Note that in all the following work, the inlet is suffix 1 and is at the inside of the rotor. The outlet is suffix 2 and is the outer edge of the rotor.

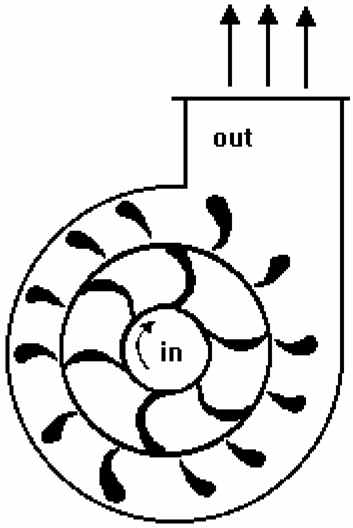


Fig. 18 Basic Design

The increase in momentum through the rotor is found as always by drawing the vector diagrams. At inlet v_1 is radial and equal to v_{r1} and so v_{w1} is zero. This is so regardless of the vane angle but there is only one angle which produces shockless entry and this must be used at the design speed.

At outlet, the shape of the vector diagram is greatly affected by the vane angle. The diagram below shows a typical vector diagram when the vane is swept backwards (referred to the vane velocity u).

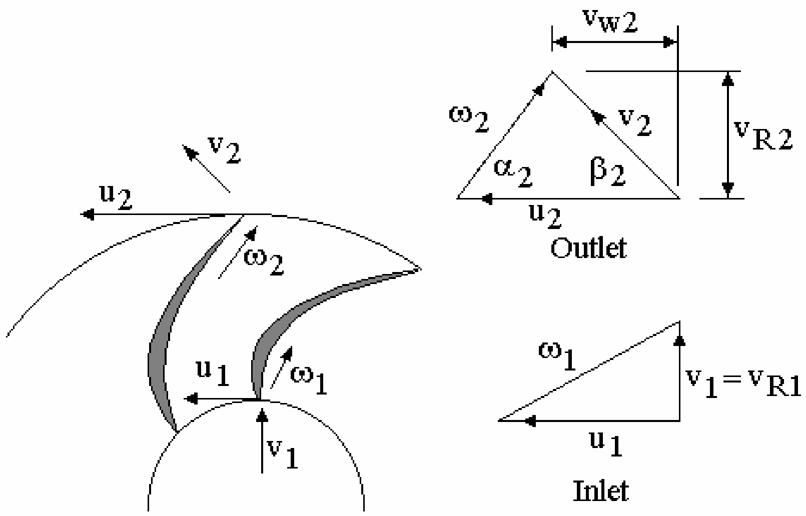


Fig. 19

v_{w2} may be found by scaling from the diagram. We can also apply trigonometry to the diagram as follows.

$$v_{w2} = u_2 - v_{r2}/\tan\alpha_2$$

$$v_{r2} = Q/\text{circumferential area} = Q/(\pi D_2 h_2 k)$$

$$u_2 = \pi N D_2$$

hence $\quad v_{w2} = u_2 - Q/(\pi D_2 h_2 k \tan \alpha_2) = u_2 - v_{r2}/\tan \alpha_2)$

DIAGRAM POWER

$$D.P.= m\Delta u v_W$$

Usually v_{W1} is zero this becomes $D.P.= m u_2 v_{W2}$

WATER POWER

$$W.P. = mg\Delta h$$

MANOMETRIC HEAD Δh_m

This is the head that would result if all the energy given to the water is converted into pressure head. It is found by equating the diagram power and water power.

$m u_2 v_{W2} = mg\Delta h_m \qquad \Delta h_m = u_2 v_{W2}/g$

MANOMETRIC EFFICIENCY η_m

$$\eta_m = W.P./D.P. = mg\Delta h/\ m u_2 v_{W2} = mg\Delta h/\ mg\Delta h_m$$

$$\eta_m = \Delta h/\Delta h_m$$

SHAFT POWER

$$\mathbf{S.P. = 2\pi NT}$$

OVERALL EFFICIENCY

$$\eta_{o/a} = \mathbf{W.P./S.P.}$$

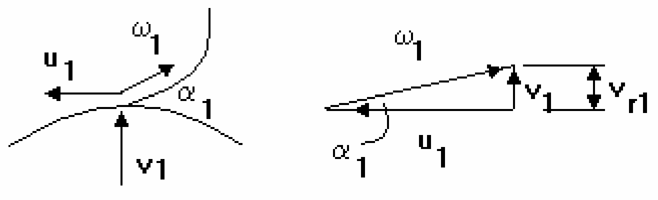
KINETIC ENERGY AT ROTOR OUTLET

$$\mathbf{K.E. = m v_2^2/2}$$

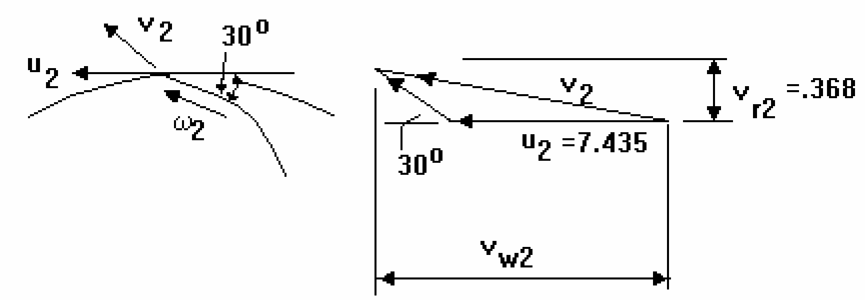
Note the energy lost is mainly in the casing and is usually expressed as a fraction of the K.E. at exit.

SELF ASSESSMENT EXERCISE No. 3

Below is the vector diagram for a centrifugal pump.



Inlet vector diagram



OUTLET VECTOR DIAGRAM

The important data follows.

The flow enters radially without shock.

Rotor outlet diameter	D_2 = 100 mm
Flow rate	Q = 0.0022 m^3/s
Density of water	1000 kg/m^3

The developed head is 5 m and the power input to the shaft is 170 Watts.

$v_{r1} = v_{r2} = 0.35$ m/s

$u_1 = 3$ m/s

$u_2 = 7.5$ m/s

Draw the vector diagrams to scale and determine the following.

a. The inlet vane angle α_1
b. The change in the velocity of whirl Δ_{vw}
c. The speed N
d. The diagram power
e. The manometric head Δh_m
f. The manometric efficiency η_m
g. The overall efficiency $\eta_{o/a}$