Every 4-connected {claw,hourglass}-free graph is 2-Hamiltonian

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Abstract

A graph G is k-hamiltonian if the graph $G - X$ is hamiltonian for any set X of k vertices of G. The claw is the complete bipartite graph $K_{1,3}$, and the hourglass is the unique graph with degree sequence $4, 2, 2, 2, 2$. We show that every 4-connected {claw,hourglass}-free graph is 2-hamiltonian. The result can be easily extended to k-hamiltonicity, implying that, for $k \geq 2$, a {claw,hourglass}-free graph G is k-hamiltonian if and only if G is $(k + 2)$-connected. This immediately implies that k-hamiltonicity is, for $k \geq 2$, polynomial-time decidable in the class of {claw,hourglass}-free graphs.

Keywords: 2-hamiltonian; closure; forbidden subgraph; claw-free; hourglass-free

1 Terminology and notation

In this paper, we generally follow the most common graph theoretical terminology and notation, and for concepts not defined here we refer the reader to [1]. Specifically, by a *graph* we always mean a simple finite undirected graph $G = (V(G), E(G))$; however, in some situations we allow multiple edges, and in this case we speak about a *multigraph*. Throughout, $d_G(x)$ denotes the *degree* of a vertex $x \in V(G)$, $N_G(x)$ the *neighborhood* of $x \in V(G)$, $O(G)$ the set of vertices of G of odd degree in G, $\kappa(G)$ the *connectivity* of G, and $\kappa'(G)$ the edge connectivity of G. By a *clique* we mean a complete subgraph, not necessarily maximal. If G_1, G_2 are graphs, we write $G_1 \simeq G_2$ if G_1 and G_2 are isomorphic, $G_1 \subset G_2$ if G_1 is a subgraph of G_2, and $G_1 \overset{\text{IND}}{\subset} G_2$ if G_1 is an induced subgraph of G_2. For $M \subset V(G)$, $\langle M\rangle_G$ denotes the induced subgraph on M, and K_M the clique on M. A vertex $x \in V(G)$ is *simplicial* if $\langle N_G(x)\rangle_G$ is a clique, and an edge $e \in E(G)$ is *pendant* if one of its vertices is of degree 1.

If $\mathcal{F}$ is a family of graphs, then a graph G is said to be *$\mathcal{F}$-free* if G does not contain a graph from $\mathcal{F}$ as an induced subgraph. In the special case when $\mathcal{F} = \{F\}$, we just say that G is *F-free*. The *claw* is the complete bipartite graph $K_{1,3}$, and the *hourglass* is the unique graph Γ with degree sequence $4, 2, 2, 2, 2$ (see Fig. 1). Whenever we list vertices of an induced subgraph, the vertices in the list are always ordered such that their degrees form a

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nonincreasing sequence (thus, e.g. the center of an induced claw or hourglass is always the first vertex of the list).

A graph G is *supereulerian* if G contains a spanning eulerian subgraph (i.e. a spanning connected subgraph with all degrees even). A spanning cycle in a graph is called a *hamiltonian cycle*, and a spanning path is called a *hamiltonian path*. If a graph G contains a hamiltonian cycle, we say that G is *hamiltonian*, and if G contains a hamiltonian (u, v)-path for any $u, v \in V(G)$, we say that G is *Hamilton-connected*. For an integer $k \geq 0$, a graph G is *k-hamiltonian* if the graph $G - X$ is hamiltonian for any set $X \subset V(G)$ with $|X| = k$, and, similarly, G is *k-Hamilton-connected* if the graph $G - X$ is Hamilton-connected for any set $X \subset V(G)$ with $|X| = k$. Recall that a k-hamiltonian graph is necessarily $(k+2)$-connected, and a k-Hamilton-connected graph is $(k+3)$-connected.

2 Introduction and main result

In claw-free graphs, an additional assumption that the graph is Γ-free, often makes difficult problems easier to handle, and allows to strengthen many results significantly.

For example, the famous conjectures by Thomassen (every 4-connected line graph is hamiltonian, [19]) and by Matthews and Sumner (every 4-connected claw-free graph is hamiltonian, [11]), which are known to be equivalent [13] but are wide open in general, become tractable under the hourglass-free assumption.

Theorem A [8]. *Every 4-connected $\{K_{1,3}, \Gamma\}$-free graph is 1-Hamilton-connected.*

As another example, one can consider pairs of forbidden subgraphs implying hamiltonicity of a 3-connected claw-free graph. While in general, the graphs $\{P_{11}, Z_8\} \cup \{N_{i,j,k} | \ i+j+k = 9\}$ are maximal graphs X implying a 3-connected $\{K_{1,3}, X\}$-free graph is hamiltonian (for the graphs Z_i, $B_{i,j}$ and $N_{i,j,k}$ see Fig. 1, for references to individual results see [17]), in the special case of Γ-free graphs, this list extends to $\{P_{20}, Z_{18}\} \cup \{N_{2i,2j,2k} | \ i + j + k = 9\}$ [17].

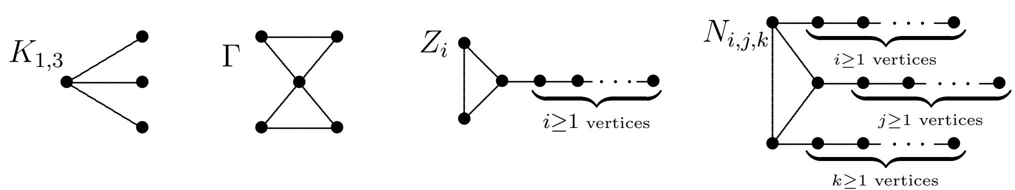

Figure 1: The claw $K_{1,3}$, the hourglass Γ, and the graphs Z_i and $N_{i,j,k}$

On the other hand, the following result on hourglass-free line graphs was proved in [20].

Theorem B [20]. *Let $L(G)$ be an hourglass-free line graph and s be an integer. If $s \geq 2$, then $L(G)$ is s-hamiltonian if and only if $\kappa(L(G)) \geq s + 2$.*

Motivated by this result, the conjectures by Thomassen and by Matthews and Sumner were strengthened in [20] as follows.

Conjecture C [20]. *Let G be a connected claw-free graph and let s be an integer. If $s \geq 2$, then G is s-hamiltonian if and only if $\kappa(G) \geq s + 2$.*

The recently developed closure technique for 2-hamiltonicity in claw-free graphs [16] provides a possibility to extend results on 2-hamiltonicity, like Theorem B, from line graphs to claw-free graphs.

The following theorem is the main result of this paper.

Theorem 1. *Let $s \geq 2$ be an integer, and let G be an $(s+2)$-connected $\{K_{1,3}, \Gamma\}$-free graph. Then G is s-hamiltonian.*

The crucial part in the proof of Theorem 1 is the case $s = 2$ given in the following theorem, from which Theorem 1 follows by an easy induction.

Theorem 2. *Let G be a 4-connected $\{K_{1,3}, \Gamma\}$-free graph. Then G is 2-hamiltonian.*

Proof of Theorem 2 is postponed to Section 4.

Proof of Theorem 1. For $s = 2$, the corollary is just Theorem 2. Thus, let $s \geq 3$, and let $X \subset V(G)$ be with $|X| = s$. Then, for any set $X' \subset X$ with $|X'| = |X| - 2$, the graph $G' = G - X'$ is 4-connected $\{K_{1,3}, \Gamma\}$-free, hence 2-hamiltonian by Theorem 2. Consequently, the graph G is s-hamiltonian. ∎

Since an s-hamiltonian graph is necessarily $(s+2)$-connected, Theorem 1 has the following immediate consequence.

Corollary 3.

(i) *Let $s \geq 2$ be an integer and let G be a $\{K_{1,3}, \Gamma\}$-free graph. Then G is s-hamiltonian if and only if G is $(s+2)$-connected.*

(ii) *For an integer $s \geq 2$, s-hamiltonicity is polynomial-time decidable in the class of $\{K_{1,3}, \Gamma\}$-free graphs*

Note that sometimes, 2-hamiltonicity is defined in a slightly stronger sense, requiring the graph $G - X$ to be hamiltonian for every set $X \subset V(G)$ with $0 \leq |X| \leq 2$ (where of course '0-hamiltonian' means 'hamiltonian'). However, in [8], it was shown that every 4-connected $\{K_{1,3}, \Gamma\}$-free graph is 1-Hamilton-connected. This immediately implies that, in our sense, every 4-connected $\{K_{1,3}, \Gamma\}$-free graph is both hamiltonian and 1-hamiltonian. Thus, our Theorem 2 is true even if 2-hamiltonicity is defined in the stronger sense.

A similar remark applies to Thorem 1 and Corollary 3.

Examples. 1. Let G be the inflation of the Petersen graph, i.e., the graph obtained from the Petersen graph by replacing each vertex with a triangle. Then G is a 3-connected $\{K_{1,3}, \Gamma\}$-free graph, but G is not 1-hamiltonian. Thus, the assumption $s \geq 2$ in Theorem 1 is sharp.

2. Let K_1, K_2, K_3 be vertex-disjoint cliques with $|V(K_j)| \geq 5$, $j = 1, 2, 3$, and let $v_i^j \in V(K_j)$, $i = 1, 2, 3, 4$, $j = 1, 2, 3$. Let G be the graph obtained from K_1, K_2, K_3 by adding the edges $v_i^1 v_i^2$ and $v_i^2 v_i^3$, $i = 1, 2, 3, 4$. Then G is 4-connected and $\{K_{1,4}, \Gamma\}$-free, but G is not 2-hamiltonian since the graph $G - \{v_3, v_4\}$ is not 1-tough hence not hamiltonian. This example shows that the assumption $K_{1,3}$ in Theorem 2 is sharp.

3 Preliminaries

In this section we summarize some known results that will be needed in the proof of Theorem 2.

3.1 Line graphs and their preimages

The *line graph* of a graph (multigraph) H is the graph $G = L(H)$ in which $V(G) = E(H)$ and two vertices of G are adjacent if and only if the corresponding edges in H have a vertex in common. Recall that every line graph is claw-free.

If G is a line graph of a graph, then it is a well-known fact that the graph H for which $G = L(H)$ is uniquely determined, with a single exception of $G = K_3$, but this is not true in line graphs of multigraphs. Thus, in line graphs of multigraphs, the "line graph preimage" is not uniquely determined. The uniqueness of the preimage can be achieved by introducing an additional requirement that simplicial vertices in the line graph correspond to pendant edges in the preimage.

Proposition D [15]. *Let G be a connected line graph of a multigraph. Then there is, up to an isomorphism, a uniquely determined multigraph H such that $G = L(H)$ and a vertex $e \in V(G)$ is simplicial in G if and only if the corresponding edge $e \in E(H)$ is a pendant edge in H.*

The graph H with the properties given in Proposition D will be called the *preimage* of a line graph G and denoted $L^{-1}(G)$ (note that for a given line graph G of a graph, $L^{-1}(G)$ and the "obvious" line graph preimage can be different). If $H = L^{-1}(G)$, and $e \in E(H)$ is the edge of H corresponding to the vertex $a \in V(G)$, we will denote $a = L(e)$ and $e = L^{-1}(a)$.

Note that a line graph $G = L(H)$ contains a graph F as an induced subgraph if and only if H contains $L^{-1}(F)$ as a (not necessarily induced) subgraph.

Krausz [9] proved the following important characterization of line graphs of graphs and of multigraphs.

Theorem E [9]. *A nonempty graph G is a line graph of a graph (of a multigraph) if and only if $V(G)$ can be covered by a system of cliques $\mathcal{K}$ such that every vertex of G is in exactly two cliques of $\mathcal{K}$ and every edge of G is in exactly one (at least one) clique of $\mathcal{K}$, respectively.*

A system of cliques $\mathcal{K} = \{K_1, ..., K_m\}$ satisfying the conditions of Theorem E will be called a *Krausz partition* of G, and the cliques $K_1, ..., K_m$ will be called *Krausz cliques*. It can be easily seen from Proposition D that a line graph G has a Krausz partition $\mathcal{K}$ such that a vertex $x \in V(G)$ is simplicial if and only if one of the two Krausz cliques containing x is of order 1. The preimage $L^{-1}(G)$ can be then obtained from such a Krausz partition $\mathcal{K}$ as the intersection graph (multigraph) of the set system $\{V(K_1), ..., V(K_m)\}$, in which the number of vertices shared by two cliques equals the multiplicity of the (multi)edge joining the corresponding vertices of $L^{-1}(G)$. Here, Krausz cliques in $G = L(H)$ correspond to the vertices of $H = L^{-1}(G)$, and the degree of a vertex of H equals the number of vertices of the corresponding Krausz clique in G. Specifically, a Krausz clique of size 3 in $G = L(H)$ will be called a *Krausz triangle*, and it corresponds to a vertex of degree 3 in H.

Note that if $G = L(H)$, where H is a graph, then, by Theorem E, any two Krausz cliques in G share at most one vertex, and any edge of G is in a uniquely determined Krausz clique. It is also easy to see that if H is triangle-free and T is a triangle in $G = L(H)$, then all edges of T are in the same (unique) Krausz clique.

A *dominating closed trail* (abbreviated DCT) in a graph H is a closed trail T (i.e., an eulerian subgraph) such that every edge of H has at least one vertex on T (note that we admit a DCT to be trivial). The following classical result by Harary and Nash-Williams shows that a DCT in a graph H corresponds to a hamiltonian cycle in $L(H)$.

Theorem F [7]. *Let H be a graph with at least 3 edges. Then $L(H)$ is hamiltonian if and only if H has a DCT.*

An edge-cut $R \subset E(H)$ of a multigraph H is *essential* if $H - R$ has at least two nontrivial components, and H is *essentially k-edge-connected* if every essential edge-cut of H has at least k edges. It is easy to see that a set $M \subset V(G)$ is a vertex-cut of a line graph G if and only if the corresponding set $L^{-1}(M) \subset E(L^{-1}(G))$ is an essential edge-cut of $L^{-1}(G)$. Consequently, a noncomplete line graph G is k-connected if and only if $L^{-1}(G)$ is essentially k-edge-connected.

If H is an essentially 3-edge-connected multigraph, then the *core* of H is the multigraph $\mathrm{co}(H)$ obtained from H by removing all pendant edges and suppressing all vertices of degree 2. It shall be noted here that this definition can be ambiguous for graphs of smaller connectivity, since then e.g. a vertex of degree 1 can be adjacent to a vertex of degree 3, and its removal creates a new vertex of degree 2, etc. These ambiguities cannot occur if G is essentially 3-edge-connected, and this is why we define the core only in this case.

Shao [18] proved the following basic properties of the core of a multigraph.

Theorem G [18]. *Let H be an essentially 3-edge-connected multigraph. Then*

(*i*) $\mathrm{co}(H)$ *is uniquely determined,*

(*ii*) $\mathrm{co}(H)$ *is 3-edge-connected,*

(*iii*) $V(\mathrm{co}(H))$ *dominates all edges of H,*

(*iv*) *if $\mathrm{co}(H)$ has a spanning closed trail, then H has a DCT,*

3.2 Closure for hamiltonicity

Let G be a graph and let $x \in V(G)$. The graph $G^*_x = (V(G), E(G) \cup \{y_1y_2 |\ y_1, y_2 \in N_G(x))$ is called the *local completion of G at x* (i.e., G^*_x is obtained from the graph G by adding all missing edges with both vertices in $N_G(x)$). A vertex $x \in V(G)$ such that $\langle N(x) \rangle_G$ is a connected (connected noncomplete) subgraph of G, is said to be *locally connected* (*eligible*), respectively. The set of all eligible vertices in G is denoted $V_{EL}(G)$. It is easy to observe that in the special case when G is a line graph and $H = L^{-1}(G)$, $x \in V(G)$ is locally connected if and only if the edge $e = L^{-1}_G(x)$ is in a triangle or in a multiedge in H, and $G^*_x = L(H|_e)$, where the graph $H|_e$ is obtained from H by contraction of e onto a vertex and replacing the created loop(s) by pendant edge(s).

In [13], it was shown that if G is claw-free then so is G^*_x, and if moreover $x \in V_{EL}(G)$, then G^*_x is hamiltonian if and only if G is hamiltonian. The *closure* $\mathrm{cl}(G)$ of a claw-free graph G was defined in [13] as the graph obtained from G by recursively performing the local completion operation at eligible vertices, as long as this is possible (more precisely: $\mathrm{cl}(G) = G_k$, where $G_1, \ldots, G_k$ is a sequence of graphs such that $G_1 = G$, $G_{i+1} = (G_i)^*_{x_i}$ for some $x_i \in V_{EL}(G_i)$, $i = 1, \ldots, k-1$, and $V_{EL}(G_k) = \emptyset$). A claw-free graph G such that $G = \mathrm{cl}(G)$ is said to be *closed*.

The following result from [13] summarizes basic properties of the closure operation.

Theorem H [13]. *For every claw-free graph G:*
(i) $\mathrm{cl}(G)$ *is uniquely determined,*
(ii) $\mathrm{cl}(G)$ *is the line graph of a triangle-free graph,*
(iii) $\mathrm{cl}(G)$ *is hamiltonian if and only if G is hamiltonian.*

3.3 Collapsible subgraphs and reduced graphs

The concept of collapsible graphs was introduced by Catlin [4]. A graph H is *collapsible* if for every subset $R \subset V(H)$ with $|R|$ even, H has a subgraph Γ_R such that $O(\Gamma_R) = R$ and $H - E(\Gamma_R)$ is connected. However, the following equivalent definition is often more useful.

Proposition I [10]. *A graph H is collapsible if and only if for every subset $R \subset V(H)$ with $|R|$ even, H has a spanning connected subgraph L_R such that $O(L_R) = R$.*

Note that the option $R = \emptyset$ in Proposition I immediately implies that every collapsible graph is supereulerian.

A simple example of a collapsible graph H is the triangle $H = C_3$: then $|V(L_R)| \in \{0, 2\}$, for $|V(L_R)| = 0$ we set $L_R = H$, and for $|V(L_R)| = 2$, L_R is a subpath of length 2 of H. In our proof, we will need the following example of a collapsible graph from [5], Theorem 2.2(iii). Here, $K_{3,3} - e$ denotes the graph obtained from the complete bipartite graph $K_{3,3}$ by removing an edge.

Fact J [5]. *The graph $K_{3,3} - e$ is collapsible.*

For a graph H and an edge subset $X \subset E(H)$, H/X denotes the graph obtained from H by contracting each edge in X and then deleting resulting loops. If F is a subgraph of H, then we use H/F for $H/E(F)$. If F is connected, and if v_F is the vertex in H/F onto which F is contracted, then F is called the *preimage* of v_F in H/F. If $F_1, F_2, \ldots, F_k$ are the list of all maximal collapsible subgraphs of H, then the graph $H' = H/(\cup_{i=1}^k F_i)$ is called the *reduction* of H. A vertex $v_F \in V(H')$ is said to be a *trivial vertex* (*nontrivial vertex*) if its preimage is a one-vertex graph (a nontrivial subgraph), respectively.

The next two theorems summarize some basic properties of collapsible graphs that will be needed in our proof.

Theorem K [4]. *Let H be a connected graph and let F be a collapsible subgraph of H. Then H is supereulerian (collapsible) if and only if H/F is supereulerian (collapsible), respectively. In particular, if H' is the reduction of H, then H is supereulerian (collapsible) if and only if H' is supereulerian (a K_1), respectively.*

For a graph H, let $F(H)$ denote the *defect of* H, i.e., the minimum number of additional edges that must be added to H to result in a graph H_1 with two edge-disjoint spanning trees. Thus $F(H) = 0$ if and only if H contains two edge-disjoint spanning trees.

Theorem L. *Let H be a connected graph.*

(i) **[4]** *If $F(H) \leq 1$, then H is collapsible if and only if $\kappa'(H) \geq 2$.*

(ii) **[5]** *If $F(H) \leq 2$, then either H is collapsible, or the reduction of H is a K_2 or a $K_{2,t}$ for some integer $t \geq 1$.*

(iii) **[6]** *For any integer $k > 0$, $\kappa'(G) \geq 2k$ if and only if for any edge subset X with $|X| \leq k$, $H - X$ has k edge-disjoint spanning trees.*

(iv) **[4]** *Let H' be the reduction of H. Then H has a DCT if and only if H' has a closed trail containing at least one vertex of each edge of H' and containing each nontrivial vertex of H'.*

3.4 Contractible subgraphs and contraction closure

For a graph H and a subgraph $F \subset H$, $H|_F$ denotes the graph obtained from H by identifying the vertices of F as a (new) vertex v_F, and by replacing the created loops by pendant edges. Note that $H|_F$ slightly differs from the contraction H/F as introduced in the previous subsection: while both can contain multiple edges, in $E(H|_F)$, the created loops are not removed but replaced by pendant edges. Consequently, we have $|E(H/F)| \leq |E(H)|$, but $|E(H|_F)| = |E(H)|$.

For a subset $X \subset V(H)$ and a partition $\mathcal{A}$ of X into subsets, $E(\mathcal{A})$ denotes the set of all edges a_1a_2 (not necessarily in H) such that a_1 and a_2 are in the same element of $\mathcal{A}$, and $H^{\mathcal{A}}$ denotes the graph with vertex set $V(H^{\mathcal{A}}) = V(H)$ and edge set $E(H^{\mathcal{A}}) = E(H) \cup E(\mathcal{A})$ (here the sets $E(H)$ and $E(\mathcal{A})$ are considered to be disjoint, i.e. if $e_1 = a_1a_2 \in E(H)$ and $e_2 = a_1a_2 \in E(\mathcal{A})$, then e_1, e_2 are parallel edges in $H^{\mathcal{A}}$).

Let F be a graph and $A \subset V(F)$. Then F is said to be A-*contractible*, if for every even subset $X \subset A$ (i.e. with $|X|$ even) and for every partition $\mathcal{A}$ of X into two-element subsets, the graph $F^{\mathcal{A}}$ has a DCT containing all vertices of A and all edges of $E(\mathcal{A})$. In particular, the case $X = \emptyset$ implies that an A-contractible graph has a DCT containing all vertices of A. Clearly, if F is A-contractible for some $A \subset V(F)$, then F is A'-contractible for any $A' \subset A$.

If H is a graph and $F \subset H$, then a vertex $x \in V(F)$ is said to be a *vertex of attachment of* F *in* H if x has a neighbor in $V(H) \setminus V(F)$. The set of all vertices of attachment of F in H is denoted $A_H(F)$. The following theorem shows that a contraction of an $A_H(F)$-contractible subgraph of a graph H does not affect the existence of a DCT.

Theorem M [14]. *Let F be a connected graph and let $A \subset V(F)$. Then F is A-contractible if and only if, for every graph H such that $F \subset H$ and $A_H(F) = A$, H has a DCT if and only if $H|_F$ has a DCT.*

Note that it was shown in [14] that if F is a collapsible graph, then F is $V(F)$-contractible (but not vice versa). Several simple examples of graphs F that are A-contractible for some $A \subset V(F)$ but not collapsible are shown in [14].

Further on, if $F \subset H$ and F is $A_H(F)$-contractible, we simply say that F is a *contractible subgraph* of H.

As shown in [14], the contractibility concept can be equivalently viewed as a closure operation on the class of line graphs. Indeed, if $F \subset H$, then the set of all edges of H, having at least one vertex in $V(F)$, corresponds in $H|_F$ to the set of all edges that contain the vertex v_F. In the line graph $G = L(H)$ this means that the set of all vertices, corresponding to the set of edges of H with at least one vertex in $V(F)$, induces a clique in $L(H|_F)$. Equivalently, $L(H|_F)$ is obtained from $L(H)$ by making the neighborhood $N_G(L(F))$ of the graph $L(F)$ complete.

We introduce a terminology similar to that in [13]. Let G be a graph and $M \overset{\text{IND}}{\subset} G$. The graph G^*_M with vertex set $V(G^*_M) = V(G)$ and edge set $E(G^*_M) = E(G) \cup \{xy|\ x, y \in N_G(M)\}$ is called the *local completion of G at M*. Obviously, if $F \subset H$, $G = L(H)$ and $M = L(F)$, then M is an induced subgraph of G, and $L(H|_F) = G^*_M$. We say that an induced subgraph M of G is *eligible* if $F = L^{-1}(M)$ is $A_H(F)$-contractible.

Thus, equivalently, eligible subgraphs of a line graph G are just line graphs of $A_H(F)$-contractible subgraphs $F \subset H = L^{-1}(G)$.

In this terminology, Theorem M has the following immediate consequence for line graphs.

Corollary N [14]. *Let G be a line graph and let $M \overset{\text{IND}}{\subset} G$ be eligible. Then G is hamiltonian if and only if G^*_M is hamiltonian.*

Now we can introduce a closure operation in a way similar to that in Subsection 3.2.

Let G be a claw-free graph. The *contraction closure of* G (or briefly the c-closure of G) is a graph $\mathrm{cl}^C(G)$ for which there is a sequence of graphs $G_1, \ldots, G_t$ such that

(i) $G_1 = \mathrm{cl}(G)$,

(ii) $G_{i+1} = (G_i)^*_M$ for some eligible subgraph $M \overset{\text{IND}}{\subset} G_i$, $i = 1, \ldots, t-1$,

(iii) $G_t = \mathrm{cl}^C(G)$ contains no eligible induced subgraph.

The following result from [14] summarizes basic properties of the c-closure.

Theorem O [14]. *Let G be a claw-free graph and let $\mathrm{cl}^C(G)$ be its c-closure. Then*

(i) *$\mathrm{cl}^C(G)$ is uniquely determined,*

(ii) *G is hamiltonian if and only if $\mathrm{cl}^C(G)$ is hamiltonian.*

A claw-free graph G such that $G = \mathrm{cl}^C(G)$ is said to be *contraction-closed*. From the previous facts it is clear that a contraction-closed graph is a line graph of a triangle-free graph and its preimage contains no contractible subgraph.

3.5 Closure for 2-hamiltonicity

The following concept, introduced in [16], will be the main tool in the proof of our main result.

Let G be a claw-free graph, and let $\bar{G}$ be a graph obtained from G by the following construction.

1. If G is not 4-connected, choose a vertex-cut $M \subset V(G)$ with $|M| \leq 3$, denote G_1, G_2 the components of $G - M$, and set $\bar{G} = (V(G), E(K_{V(G_1) \cup M}) \cup E(K_{V(G_2) \cup M}))$.
2. If G is 4-connected and 2-hamiltonian, set $\bar{G} = \mathrm{cl}(G)$.

3. If G is 4-connected and not 2-hamiltonian, then $\bar{G}$ is obtained as follows.
   a) Choose vertices x, y such that $G - \{x, y\}$ is not hamiltonian, and let $G^{(1)}$ be a graph obtained from G by the following construction:
      (*i*) set $G_0^{(1)} := G$, $i := 0$,
      (*ii*) if there is a vertex $u_i \in V(G) \setminus \{x, y\}$ that is either eligible, or nonsimplicial of degree 2 in the graph $G_i^{(1)} - \{x, y\}$, then set $G_{i+1}^{(1)} := (G_i^{(1)})^*_{u_i}$ and go to (*iii*), otherwise set $G^{(1)} := G_i^{(1)}$ and stop,
      (*iii*) set $i := i + 1$ and go to (*ii*).
   Then $G^{(1)}$ is claw-free, $G^{(1)} - \{x, y\}$ is a line graph of a triangle-free graph, and $G^{(1)} - \{x, y\}$ is not hamiltonian. Set $H^{(1)} = L^{-1}(G^{(1)} - \{x, y\})$.
   b) Let $G^{(2)}$ be a graph obtained from $G^{(1)}$ by the following construction:
      (*i*) set $G_0^{(2)} := G^{(1)}$, $H_0^{(2)} := H^{(1)}$, $i := 0$,
      (*ii*) if there is a subgraph $T_i \subset H_i^{(2)}$ such that T_i is $A_{H_i^{(2)}}(T_i)$-contractible, then set $F_i := L(T_i)$, $G_{i+1}^{(2)} := (G_i^{(2)})^*_{F_i}$, $H_{i+1}^{(2)} = L^{-1}(G_{i+1}^{(2)} - \{x, y\})$, and go to (*iii*), otherwise set $G^{(2)} := G_i^{(2)}$ and stop,
      (*iii*) set $i := i + 1$ and go to (*ii*).
   c) Set $M = \{v_1 v_2 \in E(K_{V(G)}) |\ v_1 \in \{x, y\}, v_2 \in V(G) \setminus \{v_1\}\}$, choose a maximal subset $N \subset M$ such that the graph $(V(G), E(G^{(2)}) \cup N)$ is claw-free (possibly $N = \emptyset$), and set $G^{(3)} := (V(G), E(G^{(2)}) \cup N)$.

Finally, set $\bar{G} = G^{(3)}$.

The graph $\bar{G}$ is said to be a *2h-closure* of the graph G. Note that, for a given graph G, its 2h-closure $\bar{G}$ is not necessarily uniquely determined.

Throughout the rest of the paper, we will denote $\bar{G}^- = \bar{G} - \{x, y\}$.
Immediately from the definition we can observe that

- the graph $\bar{G}$ is claw-free,
- the graph $\bar{G}^-$ is closed, contraction-closed, and all its vertices of degree 2 are simplicial.

4 Proof of Theorem 2

Let, to the contrary, G be 4-connected, $\{K_{1,3}, \Gamma\}$-free and not 2-hamiltonian, let $x, y \in V(G)$ be such that the graph $G - \{x, y\}$ is not hamiltonian, and let $\bar{G}$ be a corresponding 2h-closure of G. Set $H = L^{-1}(\bar{G})$, $e = L^{-1}(x)$, $f = L^{-1}(y)$, $\bar{G}^- = \bar{G} - \{x, y\}$, and $H^- = L^{-1}(\bar{G}^-)$ (i.e., $H^- = H - \{e, f\}$). We show that $|\{z \in V(H) |\ d_H(z) = 3\}| \leq 4$.

If every vertex of H of degree 3 is incident to e or f, then the statement is trivial. Thus, we can assume that there is a vertex $z \in V(H)$ such that $d_H(z) = 3$ and z is incident to neither e nor f (i.e., $d_{H^-}(z) = 3$). The vertex z corresponds in $\bar{G}$ to a Krausz triangle T. Denote $T = \langle\{a_1, a_2, a_3\}\rangle_{\bar{G}^-}$

We will consider some properties of the triangle T.

<u>Claim 1.</u> *No edge of T is in two Krausz cliques in $\bar{G}$.*

Proof. Let, to the contrary, K be another Krausz clique in $\bar{G}$ containing, say, the edge $a_1a_2 \in E(T)$. Set $V(K) = \{a_1, a_2, b_3, \ldots, b_k\}$. By the properties of Krausz cliques, the vertices a_1, a_2 can have neighbors only in T and K. Consequently, the set of vertices $\{a_3, b_3, \ldots, b_k\}$ is a cutset of $\bar{G}$, separating a_1, a_2 from the rest of $\bar{G}$. Since $\bar{G}$ is 4-connected, necessarily $k \geq 5$. Then at least one of the vertices $b_3, \ldots, b_k$ is in $V(\bar{G}^-)$, implying a_1 is eligible, contradicting step $3a$) of the construction of $\bar{G}$. □

Claim 2. *There are Krausz cliques K_1, K_2, K_3 in $\bar{G}$ such that $V(K_i) \cap V(T) = \{a_i\}$, $i = 1, 2, 3$, $V(K_1) \cap V(K_2) = \{x\}$ and $V(K_1) \cap V(K_3) = \{y\}$ (see Fig. 2).*

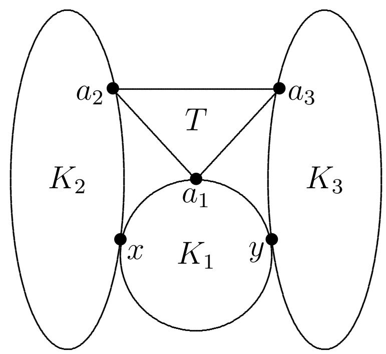

Figure 2: The Krausz cliques K_1, K_2, K_3 in Claim 2

Proof. First observe the following facts:

- any triangle in $\bar{G}^-$, created by local completion (i.e., in step $3a$) of the construction of $\bar{G}$), is a part of a larger clique in $\bar{G}$ and hence cannot be a Krausz triangle in $\bar{G}$,
- similarly, any triangle in $\bar{G}^-$, created by contraction in the preimage (i.e., in step $3b$)), is in a larger clique in $\bar{G}$ and cannot be a Krausz triangle in $\bar{G}$,
- any Krausz triangle, possibly created in step $3c$), contains x or y.

Thus, since T is a Krausz triangle not containing any of x, y, $T = \langle\{a_1, a_2, a_3\}\rangle_G$ is a triangle already in G.

Since G is 4-connected, $d_G(a_i) \geq 4$, $i = 1, 2, 3$, hence each of the vertices a_1, a_2, a_3 has at least 2 neighbors in $V(G) \setminus V(T)$. Let $b_1, b_2 \in V(G) \setminus V(T)$ be such that $a_1b_1, a_1b_2 \in E(G)$. If $a_2b_1, a_2b_2 \notin E(G)$, then $\langle\{a_1, a_2, b_1, b_2\}\rangle_G \simeq K_{1,3}$ if $b_1b_2 \notin E(G)$, or $\langle\{a_1, a_2, b_1, b_2\}\rangle_G \simeq \Gamma$ if $b_1b_2 \in E(G)$. Thus, by symmetry, we can assume that $a_2b_1 \in E(G)$.

Since $\{a_1, a_2, a_3\} \subset V(H^-)$ and a_1, a_2 cannot be eligible, necessarily $b_1 \in \{x, y\}$. Choose the notation such that $b_1 = x$. Similarly, let $c_1, c_2 \in V(G) \setminus V(T)$ be such that $a_3c_1, a_3c_2 \in E(G)$. Then, by the same argument, we have, up to a symmetry, $a_1c_1 \in E(G)$, and since a_1, a_3 cannot be eligible, $c_1 \in \{x, y\}$. However, if $c_1 = x$, then the edge a_1a_3 is in two Krausz cliques, contradicting Claim 1.

In $\bar{G}$, since the edges a_1a_2, a_1a_3 are in the same Krausz triangle T, the edges a_1x and a_2y must be in the same Krausz clique, implying $xy \in E(\bar{G})$. Denoting K_1 the Krausz clique in $\bar{G}$ containing the triangle a_1xy, K_2 the Krausz clique in $\bar{G}$ containing the edge xa_2, and K_3 the Krausz clique in $\bar{G}$ containing the edge ya_3, the claim follows. □

Claim 3. *The graph H contains at most two vertices of degree 3 that are not incident to any of the edges e, f.*

Proof. Let, to the contrary, $z_1, z_2, z_3 \in V(H^-)$ be of degree 3 in H non-incident to any of e, f, and let $T_i = \langle\{a_1^i, a_2^i, a_3^i\}\rangle_{\bar{G}}$, $i = 1, 2, 3$, be the corresponding Krausz triangles in $\bar{G}$. By Claim 2, and by the properties of Krausz cliques, there are Krausz cliques K_1, K_2, K_3 in $\bar{G}$ such that $V(K_i) \cap V(T) = \{a_i^1, a_i^2, a_i^3\}$, $i = 1, 2, 3$, $V(K_1) \cap V(K_2) = \{x\}$ and $V(K_1) \cap V(K_3) = \{y\}$ (see Fig. 3(a)). Then in $\bar{G}^-$ this graph contains as an induced subgraph the graph $L(K_{3,3})$ (see Fig. 3(b) in bold). Consequently, the graph H^- contains a subgraph isomorphic to the

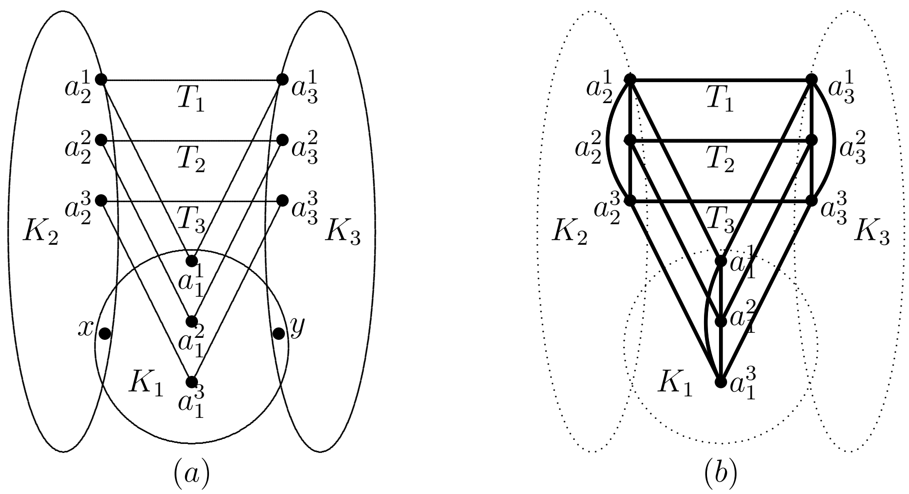

Figure 3: The Krausz triangles T_1, T_2, T_3 and the induced subgraph $L(K_{3,3})$

graph $K_{3,3}$. However, by Fact J, the graph $K_{3,3} - e$ is collapsible. Thus, the graph $K_{3,3}$ is also collapsible, hence contractible, contradicting step 3b) of the construction of $\bar{G}$. □

Claim 4. *The graph H contains at most 4 vertices of degree 3.*

Proof. By Claim 3, H contains at most 2 vertices of degree 3 non-incident to any of e, f. Thus, we have the following possibilities.

(i) *Every vertex of degree 3 in H is incident to e or f.* Then there are clearly at most 4 such vertices.

(ii) *Exactly one vertex of degree 3 in H is non-incident to any of e, f.* Let $z \in V(H^-)$ be this vertex, and let T be the corresponding Krausz triangle in $\bar{G}$. Then, by Claim 3, the only possible Krausz triangles in $\bar{G}$ are the cliques T, K_1, K_2 and K_3.

(iii) *Exactly two vertices of degree 3 in H are not incident to any of e, f.* Let $z_1, z_2 \in V(H^-)$ be these vertices, and let $T_i = \langle\{a_1^i, a_2^i, a_3^i\}\rangle_{\bar{G}}$, $i = 1, 2$, be the corresponding Krausz triangles in $\bar{G}$. Then the Krausz clique K_1 (given by Claim 3) contains vertices x, y, a_1^1 and a_1^2, hence cannot be a Krausz triangle. Thus, the only potential Krausz triangles in $\bar{G}$ are the cliques T_1, T_2, K_2 and K_3. □

Since H is essentially 4-edge-connected, a vertex of degree 3 in H cannot be adjacent to a vertex of degree 1 or 2. Consequently, a vertex $z \in V(H)$ has degree 3 in H if and only if z has degree 3 in $\mathrm{co}(H)$. This implies that, by Claim 4, $\mathrm{co}(H)$ contains at most 4 vertices of degree 3. Moreover, $\mathrm{co}(H)$ is essentially 4-edge-connected, hence $\mathrm{co}(H)$ can be turned into a 4-edge-connected graph H^+ by just adding two appropriately chosen edges. By Theorem L(iii), $\mathrm{co}(H)$ has two edge-disjoint spanning trees. Consequently, the graph $(\mathrm{co}(H))^- = \mathrm{co}(H) - \{e, f\}$ has defect $F(\mathrm{co}(H))^- \leq 2$. By Theorem L($ii$), $(\mathrm{co}(H))^-$ is either collapsible, or its reduction is K_2 or $K_{2,t}$ for some integer $t \geq 1$.

If $(\mathrm{co}(H))^-$ is collapsible, then $(\mathrm{co}(H))^-$ is supereulerian, implying $\mathrm{co}(H)$ has a spanning closed trail not containing any of the edges e, f. Thus, H has a DCT not containing any of e, f, hence H^- has a DCT, implying $L(H^-) = \bar{G}^-$ is hamiltonian, a contradiction. Thus, the reduction of $(\mathrm{co}(H))^-$ is K_2 or $K_{2,t}$ for some integer $t \geq 1$. Let H_R denote the reduction of $(\mathrm{co}(H))^-$.

Let first $H_R = K_2$, and set $V(H_R) = \{v_1, v_2\}$. Then one of the vertices of H_R, say, v_2, is trivial, for otherwise $\mathrm{co}(H)$ cannot be essentially 4-edge-connected. However, then the trivial closed trail with vertex set $\{v_1\}$ satisfies the assumptions of Theorem L(iv), hence H^- has a DCT, a contradiction.

Thus, we have $H_R = K_{2,t}$ for some integer $t \geq 1$. Set $V(K_{2,t}) = (\{v_1, v_2\}, \{u_1, \ldots, u_t\})$. If t is even, then $u_1v_1u_2v_2 \ldots u_{t-1}v_1u_tv_2u_1$ is a closed trail satisfying the assumptions of Theorem L(iv), hence H^- has a DCT, a contradiction. Thus, t is odd. The case $t = 1$ immediately contradicts the fact that $\mathrm{co}(H)$ is essentially 4-edge-connected. Thus, t is odd and $t \geq 3$. Observe that each of the vertices $u_1, \ldots, u_t$ is of degree 2. Since $\delta(\mathrm{co}(H)) \geq 3$, each of the vertices $u_1, \ldots u_t$ is incident to at least one of the edges e, f, which immediately implies $t = 3$. Thus, we have that $(\mathrm{co}(H))^-$ is reducible to $K_{2,3}$.

Again, since $\delta(\mathrm{co}(H)) \geq 3$, each of the vertices u_1, u_2, u_3 is incident in $\mathrm{co}(H)$ to at least one of the edges e, f. By a simple counting we see that at least one of u_1, u_2, u_3 is incident to exactly one of e, f and is therefore of degree 3 in $\mathrm{co}(H)$. Choose the notation such that $d_{\mathrm{co}(H)}(u_1) = 3$. Then u_1 is a trivial vertex since $\mathrm{co}(H)$ is essentially 4-edge-connected. Then the cycle $v_1u_2v_2u_3v_1$ is a closed trail containing at least one vertex of each edge and each nontrivial vertex of the $K_{2,3}$. By Theorem L(iv), H^- has a DCT, a contradiction. ■

5 Concluding remarks

The following concepts were introduced in [12] (see also [2, 3]). If $\mathcal{C}$ is a subclass of the class of claw-free graphs, then $\mathcal{C}$ is said to be *stable* if, for any $G \in \mathcal{C}$, the local completion of G at any vertex is also in $\mathcal{C}$. If $\mathfrak{cl}$ is a closure operation that turns a claw-free graph into a line graph by a series of local completions and $\mathcal{C}$ is stable, then $\mathfrak{cl}(G) \in \mathcal{C}$ for any $G \in \mathcal{C}$.

It is not difficult to observe that the class of $\{K_{1,3}, \Gamma\}$-free graphs is not stable in this sense (see e.g. [12]), however, it can be shown [3] that if G is $\{K_{1,3}, \Gamma\}$-free, then $\mathrm{cl}(G)$ (where cl is the closure for hamiltonicity mentioned in Section 3.2) is still $\{K_{1,3}, \Gamma\}$-free. Thus, we say that a class $\mathcal{C}$ is *weakly stable under a closure operation* $\mathfrak{cl}$ if $\mathfrak{cl}(G) \in \mathcal{C}$ for any $G \in \mathcal{C}$. It was shown in [3] that the class of $\{K_{1,3}, \Gamma\}$-free graphs is weakly stable under cl (although it is not stable).

If a similar result was true also for the 2h-closure, then Theorem 2 would be an immediate consequence of Theorem B. However, it is not difficult to see that, in general, it might happen that any 2h-closure of a $\{K_{1,3}, \Gamma\}$-free graph contains an induced Γ (such induced hourglasses are typically created e.g. in Step 1 of the construction of $\bar{G}$, or in Step 3a) if the vertex u_i is of degree 2). For this reason, we are convinced that such a shortcut is not possible, and a direct proof of Theorem 2 is necessary.

On the other hand, having proved Theorem 2 independently, we observe that if G is 4-connected $\{K_{1,3}, \Gamma\}$-free, then, by Theorem 2, G is 2-hamiltonian. Hence Step 2 of the construction of $\bar{G}$ applies, and we have $\bar{G} = \mathrm{cl}(G)$. This means that, after having proved Theorem 2 independently, we observe that Γ is weakly stable under the 2h-closure in the class of 4-connected $\{K_{1,3}, \Gamma\}$-free graphs.

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